

Numerical optimization and optimal control for dynamical systems [146442]

Additional Exercises

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Appendices

Contents

This volume collects the exercises, worked examples and case studies of the course *Numerical optimization and optimal control for dynamical systems*. The theory they rest on is in the companion volume, the course notes, whose parts and chapter names this book mirrors.

Each part can be read on its own once the corresponding part of the notes has been covered. The examples that are needed to develop the exposition have been left in the notes and are not repeated here.

- **Constrained Minimization**
  - KKT Worked Examples
- **Differential-Algebraic Equations**
  - Case Studies
    - Nonlinear Reduction
    - 3D Pendulum
    - Double Pendulum
    - Circuit DAE
    - Index-5 Crane
  - DAE Exercises
- **Calculus of Variations**
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Constrained Minimization

1 KKT Worked Examples 7

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1.13 Example 16: A Continuum of Global Minimizers and Failure of Strict Second-Order Sufficiency

Exercises and worked examples on constrained minimization: Lagrange multipliers, the Karush–Kuhn–Tucker conditions, and the second-order tests that go with them. The corresponding theory is in the part of the same name of the course notes.

1. KKT Worked Examples

1.1 Example 1: Detailed Example of Constrained Minimization

In this example, we will minimize the function

$$\begin{aligned} \text{minimize:} \quad & f(x, y) = x + y, \\ \text{subject to the constraint:} \quad & h(x, y) = x^2 + y^2 - 1 = 0, \end{aligned}$$

where $h(x, y) = 0$ describes a unit circle. We will approach this problem using two methods:

1. **Parametrization with $x = \cos(t)$ and $y = \sin(t)$.**
2. **Lagrange Multipliers.**

1.1.1 Parametric Solution

The constraint $h(x, y) = x^2 + y^2 - 1$ represents a unit circle. We can parametrize the constraint as:

$$x(t) = \cos(t), \quad y(t) = \sin(t), \quad t \in [0, 2\pi].$$

Substituting the parametrization into the objective function yields:

$$f(\cos(t), \sin(t)) = \cos(t) + \sin(t).$$

To find the critical points, we can differentiate this parametric function and analyze it:

1.1.1.1 Critical Points

The maximum and minimum values of f can be found by solving for t:

- The function $f(\cos(t), \sin(t))$ achieves its extrema when:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cos(t) + \sin(t)) = -\sin(t) + \cos(t) = 0.$$

This gives us:

$$\sin(t) = \cos(t) \quad \Rightarrow \quad t = \frac{\pi}{4} + k\pi, \; k \in \mathbb{Z}.$$

1.1.1.2 Function Values

Evaluating the function at these points:

- At $t = \frac{5\pi}{4}$ (minimum):

$$\left.\begin{aligned} x &= \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \\ y &= \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \end{aligned}\right\} \quad \Longrightarrow \quad f(x, y) = -\sqrt{2}.$$

- At $t = \frac{\pi}{4}$ (maximum):

$$\left.\begin{aligned} x &= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \\ y &= \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \end{aligned}\right\} \quad \Longrightarrow \quad f(x, y) = \sqrt{2}.$$

1.1.1.3 Visualization

Let's visualize the unit circle and the level sets of the objective function.

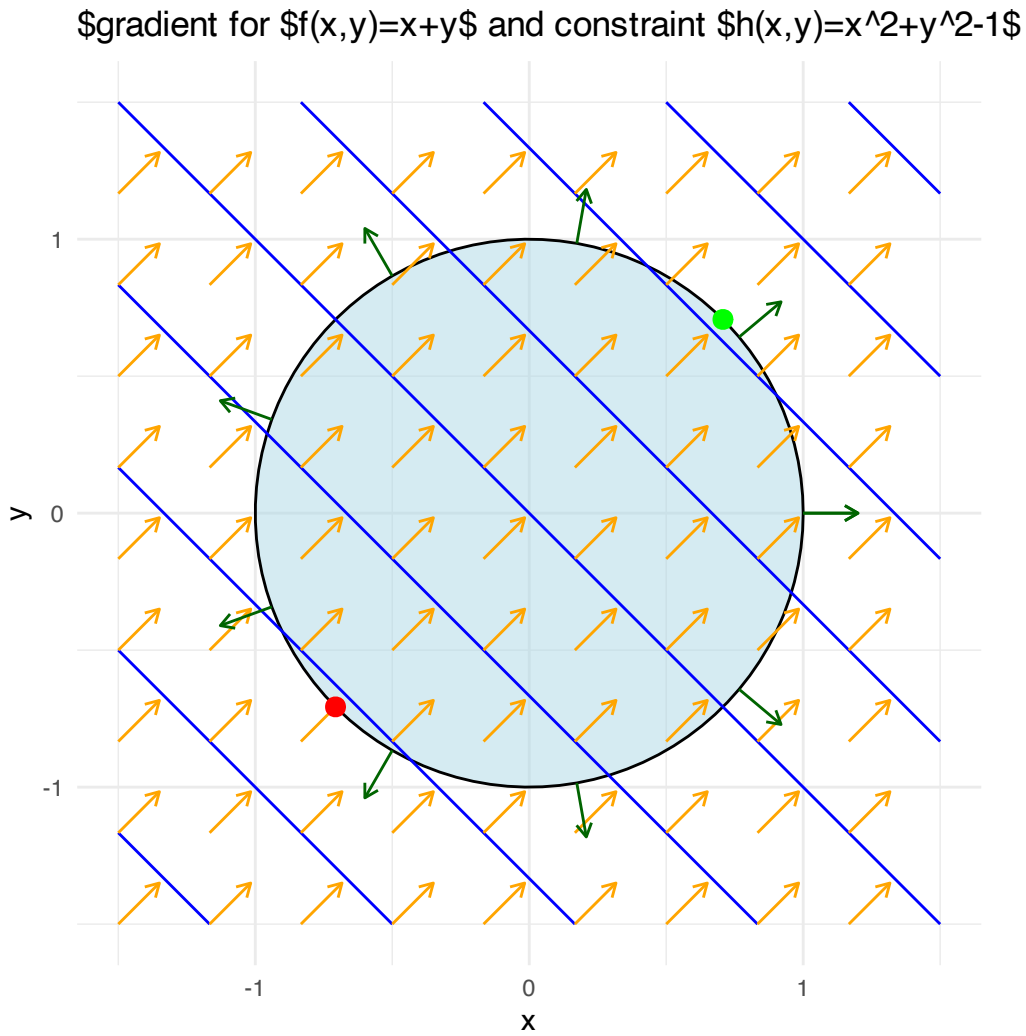


Figure 1.1: Objective level sets, gradient field, and the unit-circle constraint.

The plot illustrates that the minimum and maximum points occur where the gradient vector field of the function $f(x, y) = x + y$ is aligned with the normal vectors of the constraint

curve $h(x, y) = x^2 + y^2 - 1$. This alignment indicates that the gradient is parallel to the constraint's normal, satisfying the first order necessary condition.

1.1.2 Lagrange Multipliers Method

Now, we will apply the method of Lagrange multipliers to our constrained optimization problem. We define the Lagrangian as follows:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda h(x, y) = x + y - \lambda(x^2 + y^2 - 1).$$

1.1.2.1 First-Order Conditions

To find the extrema, we take the gradients of the Lagrangian with respect to x, y, and λ. This leads to the first-order conditions:

1. For x:
$$\nabla_x \mathcal{L} = 1 - 2\lambda x = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2x}.$$
2. For y:
$$\nabla_y \mathcal{L} = 1 - 2\lambda y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y}.$$
3. The constraint condition:
$$h(x, y) = x^2 + y^2 - 1 = 0.$$

1.1.2.2 Solving the nonlinear system

Setting the two expressions for λ equal gives us:

$$\frac{1}{2x} = \frac{1}{2y} \quad \Rightarrow \quad x = y.$$

Next, substituting $x = y$ into the constraint:

$$2x^2 = 1 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}.$$

1.1.2.3 Second-Order Conditions

To determine whether the critical points are minima or maxima, we need to evaluate the second-order conditions.

1. **Second-Order Necessary Condition**: The Hessian of the Lagrangian, $\nabla^2_{(x,y)}\mathcal{L}(x, y, \lambda)$, must be **positive semidefinite** in the null space of the constraint gradients.
2. **Second-Order Sufficient Condition**: The Hessian of the Lagrangian, $\nabla^2_{(x,y)}\mathcal{L}(x, y, \lambda)$, must be **positive definite** in the null space of the constraint gradients.

1.1.2.4 Computing the Hessian

The Hessian matrix of the Lagrangian with respect to x and y is given by:

$$\nabla^2 \mathcal{L}(x, y, \lambda) = \begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial^2 \mathcal{L}}{\partial y \partial x} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{bmatrix}.$$

1.1.2.5 Kernel of the Constraint

The kernel of the constraint

$$h(x, y) = 0$$

consists of the vectors z that satisfy

$$\nabla h(x, y) \cdot z = 0.$$

The gradient of the constraint is:

$$\nabla h(x, y) = [2x, 2y]$$

and it is easy to verify that the kernel of the constraint gradient can be spanned by vector

$$z = \begin{pmatrix} -y \\ x \end{pmatrix}$$

1.1.2.6 Check qualification

The equality-constraint gradient is nonzero at both candidates, hence LICQ holds.

1.1.2.7 Evaluating the Hessian at Critical Points

1. At point $(x^\star, y^\star) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ the multiplier $\lambda = -\frac{1}{\sqrt{2}}$. The kernel of $h(x, y)$ is spanned by

$$z = t \begin{pmatrix} y^\star \\ -x^\star \end{pmatrix} = \frac{t}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \ker\left(\nabla h(x^\star, y^\star)^\top\right)$$

Thus the Hessian and second order condition becomes:

$$\nabla^2_{(x,y)} \mathcal{L} = \sqrt{2}\, I, \qquad z^\top \nabla^2_{(x,y)} \mathcal{L} z \quad \Longrightarrow$$

$$\frac{t^2}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \left(\sqrt{2}\, I\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sqrt{2}\, t^2 > 0,$$

for all $t \neq 0$. Therefore, this point is a **strict local minimum**.

2. At point $(x^\star, y^\star) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ the multiplier $\lambda = \frac{1}{\sqrt{2}}$.
Thus the Hessian and second order condition becomes:

$$\nabla^2_{(x,y)} \mathcal{L} = -\sqrt{2}\, I, \qquad z^\top \nabla^2_{(x,y)} \mathcal{L} z \quad \Longrightarrow$$

$$\frac{t^2}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \left(-\sqrt{2}\, I\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\sqrt{2}\, t^2 < 0, \qquad \forall t \neq 0$$

Therefore, this point is a **strict local maximum**.

1.1.3 Conclusion

Both methods yield the same results:

- The **minimum** occurs at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ with $f(x, y) = -\sqrt{2}$.
- The **maximum** occurs at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ with $f(x, y) = \sqrt{2}$.

1.2 Example 2: Detailed Example of KKT

In this example, we will minimize the function

$$\begin{aligned} \text{minimize:} \quad & f(x, y, z) = x^2 + y + z, \\ \text{equality constraint:} \quad & x + y^2 + z^2 = 1, \\ \text{and inequalities:} \quad & z \leq 1, \quad x \geq 0, \end{aligned}$$

1.2.0.1 Rewriting in Standard Form

We can rewrite the problem in a more standard form:

$$\begin{aligned} \text{minimize:} \quad & f(x,y,z) = x^2 + y + z, \\ \text{equality constraint:} \quad & h(x,y,z) = x + y^2 + z^2 - 1 = 0, \\ \text{and inequalities:} \quad & g_1(x,y,z) = 1 - z \geq 0, \\ & g_2(x,y,z) = x \geq 0, \end{aligned}$$

Here, $h(x,y,z) = 0$ defines the paraboloid $x = 1 - y^2 - z^2$. Notice also that, once $x \geq 0$ and the equality constraint hold, one has $y^2 + z^2 \leq 1$, hence automatically $z \leq 1$. Therefore $g_1 = 1 - z \geq 0$ is a **redundant constraint**. We keep it in the KKT system because it is useful for illustrating inactive multipliers.

1.2.0.2 Lagrangian Function

We define the Lagrangian function as:

$$\mathcal{L}(x,y,z,\lambda,\mu_1,\mu_2) = f(x,y,z) - \lambda h(x,y,z) - \mu_1 g_1(x,y,z) - \mu_2 g_2(x,y,z)$$

Substituting the expressions for f, h, g_1, and g_2:

$$\mathcal{L}(x,y,z,\lambda,\mu_1,\mu_2) = x^2 + y + z - \lambda(x + y^2 + z^2 - 1) - \mu_1(1 - z) - \mu_2 x$$

1.2.0.3 First-Order Conditions

To satisfy the first-order conditions, we compute the partial derivatives of $\mathcal{L}$:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 2x - \lambda - \mu_2 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = 1 - 2\lambda y = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z} = 1 - 2\lambda z + \mu_1 = 0 \end{cases}$$

The KKT conditions also require the original constraints to hold:

$$\begin{cases} x + y^2 + z^2 - 1 = 0 \\ \mu_1(1 - z) = 0 \\ \mu_2 x = 0 \end{cases}$$

with primal feasibility $1 - z \geq 0$, $x \geq 0$ and dual feasibility $\mu_1, \mu_2 \geq 0$.

We now proceed with different cases based on the values of μ_1 and μ_2

- **Case 1:** $\mu_1 = \mu_2 = 0$

  In this case, the first-order conditions simplify to:

$$\begin{cases} 2x - \lambda = 0 \\ 1 - 2\lambda y = 0 \\ 1 - 2\lambda z = 0 \\ x + y^2 + z^2 - 1 = 0 \end{cases} \implies \begin{cases} x = \lambda/2 \\ y = 1/(2\lambda) \\ z = 1/(2\lambda) \\ \lambda/2 + 2/(2\lambda)^2 - 1 = 0 \end{cases} \implies \lambda^3 - 2\lambda^2 + 1 = 0$$

  solving the cubic (with Maple) the real roots are

$$\lambda = 1, \ \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}$$

From these values, we compute the corresponding x, y, and z:

$$\mu_1 = \mu_2 = 0, \quad \begin{cases} x = y = z = 1/2, & \lambda = 1 \\ x = (1+\sqrt{5})/4, & y = z = 1/(\sqrt{5}+1), \quad \lambda = (\sqrt{5}+1)/2 \\ x = (1-\sqrt{5})/4 < 0, & y = z = 1/(1-\sqrt{5}), \quad \lambda = (1-\sqrt{5})/2 \end{cases}$$

the last solution must be discarded

- **Case 2:** $\mu_1 = 0$**, and** $\mu_2 > 0$

  In this case, $x = 0$, and the system becomes:

$$\begin{cases} 2x - \lambda - \mu_2 = 0 \\ 1 - 2\lambda y = 0 \\ 1 - 2\lambda z = 0 \\ x + y^2 + z^2 - 1 = 0 \\ x = 0 \end{cases} \implies \begin{cases} \lambda = -\mu_2 \\ y = 1/(2\lambda) = -1/(2\mu_2) \\ z = 1/(2\lambda) = -1/(2\mu_2) \\ 2/(2\mu_2)^2 - 1 = 0 \\ x = 0 \end{cases} \implies \mu_2^2 = \frac{1}{2}$$

  the positive root is $\mu_2 = 1/\sqrt{2}$ so that

$$x = 0, \quad y = z = -\frac{1}{2\mu_2} = -\frac{\sqrt{2}}{2}, \quad \lambda = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \mu_1 = 0, \quad \mu_2 = \frac{1}{\sqrt{2}}$$

- **Case 3:** $\mu_2 = 0$**, and** $\mu_1 > 0$

$$\begin{cases} 2x - \lambda = 0 \\ 1 - 2\lambda y = 0 \\ 1 - 2\lambda z + \mu_1 = 0 \\ x + y^2 + z^2 - 1 = 0 \\ z = 1 \end{cases} \implies \begin{cases} 2x - \lambda = 0 \\ 1 - 2\lambda y = 0 \\ 1 - 2\lambda + \mu_1 = 0 \\ x + y^2 = 0 \\ z = 1 \end{cases} \implies \begin{cases} x = \lambda/2 \\ y = 1/(2\lambda) \\ \mu_1 = 2\lambda - 1 \\ x + y^2 = 0 \\ z = 1 \end{cases}$$

  so that the real root of λ is

$$\frac{\lambda}{2} + \frac{1}{4\lambda^2} = 0 \implies 2\lambda^3 + 1 = 0 \implies \lambda = -2^{-1/3} \implies$$

  and

$$x = -2^{-4/3} < 0, \quad y = 2^{-2/3}, \quad z = 1, \quad \lambda = -2^{-1/3}, \quad \mu_1 = 2^{2/3} - 1 > 0, \quad \mu_2 = 0$$

  the solution must be discarded.

- **Case 4:** $\mu_2 > 0$ **and** $\mu_1 > 0$

$$\begin{cases} 2x - \lambda - \mu_2 = 0 \\ 1 - 2\lambda y = 0 \\ 1 - 2\lambda z + \mu_1 = 0 \\ x + y^2 + z^2 - 1 = 0 \\ z = 1 \\ x = 0 \end{cases} \implies \begin{cases} -\lambda - \mu_2 = 0 \\ 1 - 2\lambda y = 0 \\ 1 - 2\lambda + \mu_1 = 0 \\ y^2 = 0 \\ z = 1 \\ x = 0 \end{cases} \implies 1 - 2\lambda \cdot 0 = 0, \quad \text{no solution}$$

1.2.0.4 All the solutions

x	y	z	λ	μ_1	μ_2
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0
$\frac{1+\sqrt{5}}{4}$	$\frac{1}{\sqrt{5}+1}$	$\frac{1}{\sqrt{5}+1}$	$\frac{\sqrt{5}+1}{2}$	0	0
0	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$

1.2.1 Study of solution 1

$x = y = z = \frac{1}{2}, \lambda = 1, \mu_1 = \mu_2 = 0$

1.2.1.1 Compute active constraints

$$g_1(x, y, z) = 1 - z = 1 - \frac{1}{2} = \frac{1}{2} > 0, \qquad \text{(not-active)}$$

$$g_2(x, y, z) = x = \frac{1}{2} > 0, \qquad \text{(not-active)}$$

Gradient of active constraints

$$\nabla h(x, y, z) = \nabla(x + y^2 + z^2 - 1) = [1, 2y, 2z] = [1, 1, 1]$$

find the kernel of the active constraint

$$[1,1,1]\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z_1 + z_2 + z_3 \implies \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{K^a} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

1.2.1.2 Check qualification

The equality-constraint gradient is nonzero, hence LICQ holds.

1.2.1.3 Compute Hessian and reduced Hessian

$$\nabla^2_{(x,y,z)}\mathcal{L} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \implies \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

reduced Hessian

$$\begin{aligned} K^{\top,a}\nabla^2_{(x,y,z)}K^a &= \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ -2 & 0 \\ 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \end{aligned}$$

The determinant of the reduced Hessian is

$$\det\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = -4 < 0,$$

so the reduced Hessian is indefinite. Hence the second-order necessary condition fails and the point is **not a local minimum**.

1.2.2 Study of solution 2

$x = \frac{\sqrt{5}+1}{4}$, $y = z = \frac{\sqrt{5}-1}{4}$, $\lambda = \frac{\sqrt{5}+1}{2}$, $\mu_1 = \mu_2 = 0$

1.2.2.1 Compute active constraints

$$g_1(x,y,z) = 1 - z = 1 - \frac{\sqrt{5}-1}{4} = \frac{5-\sqrt{5}}{4} > 0, \qquad \text{(not-active)}$$

$$g_2(x,y,z) = x = \frac{\sqrt{5}+1}{4} > 0, \qquad \text{(not-active)}$$

Gradient of active constraints

$$\nabla h(x,y,z) = \begin{pmatrix} 1 & 2y & 2z \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

find the kernel of the active constraint

$$\begin{pmatrix} 1 & \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z_1 + \frac{\sqrt{5}-1}{2}(z_2 + z_3)$$

$$\Downarrow$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2}(\alpha+\beta) \\ \alpha \\ \beta \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{K^a} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

1.2.2.2 Check qualification

The equality-constraint gradient is nonzero, hence LICQ holds.

1.2.2.3 Compute Hessian and reduced Hessian

$$\nabla^2_{(x,y,z)} \mathcal{L} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -(\sqrt{5}+1) & 0 \\ 0 & 0 & -(\sqrt{5}+1) \end{pmatrix}$$

reduced Hessian. Let

$$a = \frac{1-\sqrt{5}}{2}, \qquad K^a = \begin{pmatrix} a & a \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$(K^a)^\top \nabla^2_{(x,y,z)} \mathcal{L}\, K^a = \begin{pmatrix} 2-2\sqrt{5} & 3-\sqrt{5} \\ 3-\sqrt{5} & 2-2\sqrt{5} \end{pmatrix}.$$

Already the first leading principal minor is

$$2 - 2\sqrt{5} < 0.$$

Therefore the reduced Hessian is not positive semidefinite. The second-order necessary condition for a local minimum fails, and this point is **not a local minimum**.

1.2.3 Study of solution 3

$x = 0$, $y = z = \lambda = -\frac{\sqrt{2}}{2}$, $\mu_1 = 0$, $\mu_2 = \frac{\sqrt{2}}{2}$

1.2.3.1 Compute active constraints

$$g_1(x,y,z) = 1 - z = 1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2} > 0, \qquad \text{(not-active)}$$
$$g_2(x,y,z) = x = 0, \qquad \text{(active)}$$

Gradient of active constraints

$$\begin{pmatrix} \nabla h(x,y,z) \\ \nabla g_2(x,y,z) \end{pmatrix} = \begin{pmatrix} 1 & 2y & 2z \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix}$$

find the kernel of the active constraint

$$\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 - \sqrt{2}(z_2+z_3) \\ z_1 \end{pmatrix}$$
$$\Downarrow$$
$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ -\alpha \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{K^a} \alpha \tag{1.1}$$

1.2.3.2 Check qualification

The gradients in Equation 1.1 are linearly independent, hence LICQ holds.

1.2.3.3 Compute Hessian and reduced Hessian

$$\nabla^2_{(x,y,z)} \mathcal{L} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

reduced Hessian

$$\begin{aligned} \boldsymbol{K}^{\top,a} \nabla^2_{(x,y,z)} \boldsymbol{K}^a &= \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2} \\ -\sqrt{2} \end{pmatrix} \\ &= 2\sqrt{2} > 0 \end{aligned}$$

Since the reduced Hessian is the positive scalar $2\sqrt{2}$, the second-order sufficient condition holds. Thus the point is a **strict local minimum**. In fact it is the unique global minimizer: feasibility and $x \geq 0$ give $y^2 + z^2 \leq 1$, and

$$f = x^2 + y + z \geq y + z \geq -\sqrt{2},$$

with equality only at $x = 0$ and $y = z = -1/\sqrt{2}$.

Example 3

Consider the problem

$$\begin{aligned} \text{minimize:} \quad & f(x, y) = y \\ \text{inequalities:} \quad & g_1(x, y) = x^2 + y^2 - 1 \geq 0, \\ & g_2(x, y) = 1 - x^2 - \frac{1}{4}y^2 \geq 0, \end{aligned}$$

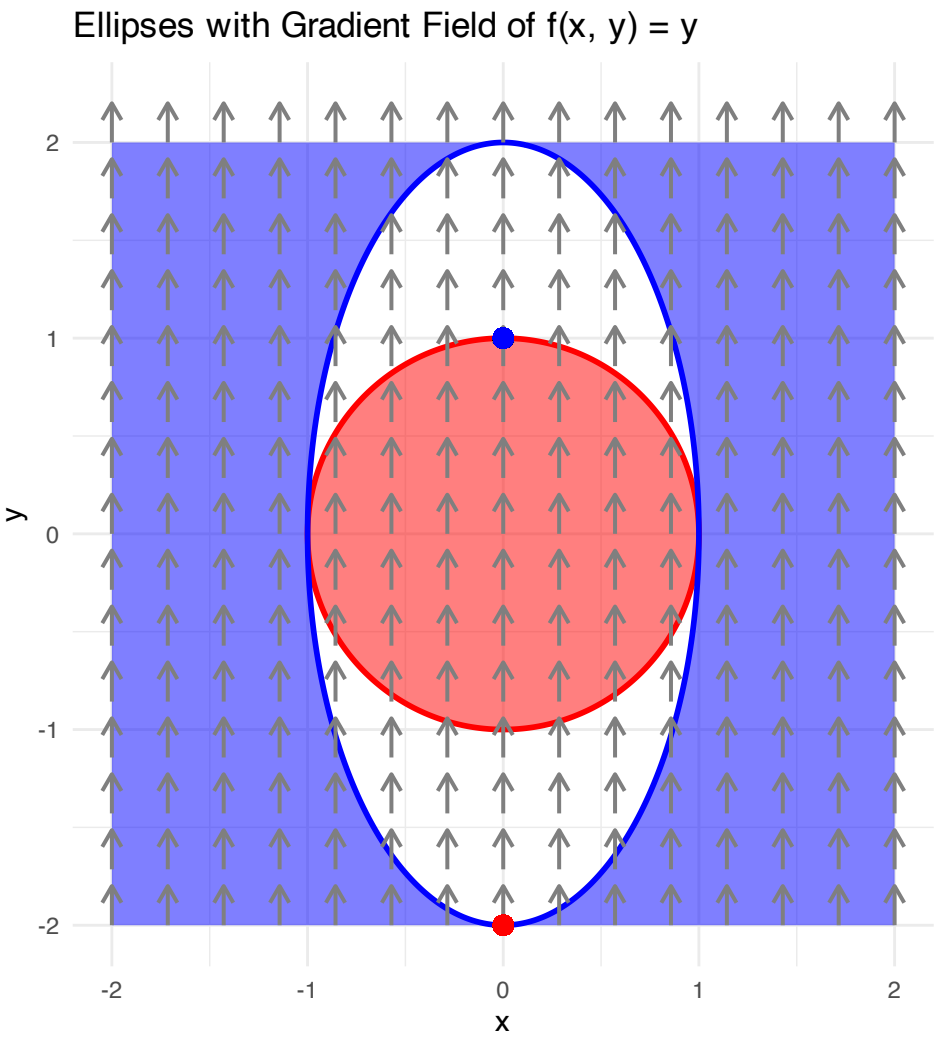


Figure 1.2: Feasible region and constrained minimum for the two-constraint problem.

The white area is the feasible region of the problem. The red point is the constrained minimum.

1.2.4 Lagrangian Function

We define the Lagrangian function as follows:

$$\mathcal{L}(x, y, \mu_1, \mu_2) = y - \mu_1(x^2 + y^2 - 1) - \mu_2 \left(1 - x^2 - \frac{1}{4}y^2\right)$$

1.2.5 First-Order Conditions

The first-order conditions derived from the Lagrangian are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 2x(\mu_2 - \mu_1), \\
\frac{\partial \mathcal{L}}{\partial y} &= 1 + y\left(\frac{1}{2}\mu_2 - 2\mu_1\right), \\
0 &= \mu_1(x^2 + y^2 - 1), \\
0 &= \mu_2\left(1 - x^2 - \frac{1}{4}y^2\right).
\end{aligned}$$

1.2.6 Solutions of First-Order Conditions

The solutions to the first-order conditions yield the following critical points:

$$\begin{aligned}
&x = 0, \quad y = 1, \quad \mu_1 = \frac{1}{2}, \quad \mu_2 = 0, \\
&x = 0, \quad y = -2, \quad \mu_1 = 0, \quad \mu_2 = 1.
\end{aligned}$$

1.2.7 Analysis of First Candidate

1.2.7.1 Compute Active Constraints

We evaluate the constraints at the first candidate $(x, y) = (0, 1)$:

$$\begin{aligned}
g_1(x, y) &= x^2 + y^2 - 1 = 0^2 + 1^2 - 1 = 0, && \text{(active)} \\
g_2(x, y) &= 1 - x^2 - \frac{1}{4}y^2 = 1 - 0^2 - \frac{1}{4}1^2 = \frac{3}{4} > 0, && \text{(not-active)}
\end{aligned}$$

1.2.7.2 Gradient of Active Constraints

The gradient of the active constraint g_1 is computed as follows:

$$\nabla g_1(x, y) = [2x, 2y] = [0, 2].$$

1.2.7.3 Check qualification

The single active gradient $\nabla g_1(x, y) = [0, 2]$ is nonzero, so LICQ holds.

1.2.7.4 Kernel of the Gradient of Active Constraints

The kernel of the gradient can be expressed as:

$$[0, 2]\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2z_2 = 0 \implies \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{K^a} \alpha.$$

1.2.7.5 Reduced Hessian

The reduced Hessian is given by:

$$\begin{aligned}
\nabla^2_{(x,y)}\mathcal{L}(x, y, \mu_1, \mu_2) &= \begin{pmatrix} 2(\mu_2 - \mu_1) & 0 \\ 0 & \frac{1}{2}(\mu_2 - 4\mu_1) \end{pmatrix} \\
&\implies \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

Evaluating the quadratic form yields:

$$K^{a,\top}\nabla^2_{(x,y)}\mathcal{L}(x, y, \mu_1, \mu_2)K^a = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1.$$

Because $\mu_1 = 1/2 > 0$, the critical cone is the tangent line $z_2 = 0$. The reduced Hessian equals -1 on that line, so the second-order necessary condition fails and this point is not a local minimum.

1.2.8 Analysis of Second Candidate

1.2.8.1 Compute Active Constraints

Next, we evaluate the constraints at the second candidate $(x, y) = (0, -2)$:

$$g_1(x, y) = x^2 + y^2 - 1 = 0^2 + (-2)^2 - 1 = 3 > 0, \quad \text{(not-active)}$$

$$g_2(x, y) = 1 - x^2 - \frac{1}{4}y^2 = 1 - 0^2 - \frac{1}{4}(-2)^2 = 0, \qquad \text{(active)}$$

1.2.8.2 Gradient of Active Constraints

The gradient of the active constraint g_2 is computed as:

$$\nabla g_2(x, y) = [-2x, -\tfrac{1}{2}y] = [0, 1].$$

1.2.8.3 Check qualification

The single active gradient $\nabla g_2(x, y) = [0, 1]$ is nonzero, so LICQ holds.

1.2.8.4 Kernel of the Gradient of Active Constraints

The kernel of the gradient can be expressed as:

$$[0, 1]\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_2 = 0 \implies \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{K^a} \alpha.$$

1.2.8.5 Reduced Hessian

The reduced Hessian is given by:

$$\nabla^2_{(x,y)}\mathcal{L}(x, y, \mu_1, \mu_2) = \begin{pmatrix} 2(\mu_2 - \mu_1) & 0 \\ 0 & \frac{1}{2}(\mu_2 - 4\mu_1) \end{pmatrix} \implies \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Evaluating the quadratic form yields:

$$K^{a,\top}\nabla^2_{(x,y)}\mathcal{L}(x, y, \mu_1, \mu_2)K^a = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2.$$

Because the active multiplier is $\mu_2 = 1 > 0$, the critical cone equals this tangent line. The reduced Hessian is positive definite on it, so the second-order sufficient condition holds and $(0, -2)$ is a **strict local minimum**.

Example 4

Consider the problem

$$\begin{aligned} \text{minimize:} \quad & f(x, y) = x \\ \text{inequalities:} \quad & g_1(x, y) = x^2 + y^2 - 1 \geq 0, \\ & g_2(x, y) = 1 - x^2 - \frac{1}{4}y^2 \geq 0, \end{aligned}$$

The white area is the feasible region of the problem. The red point is the constrained minimum.

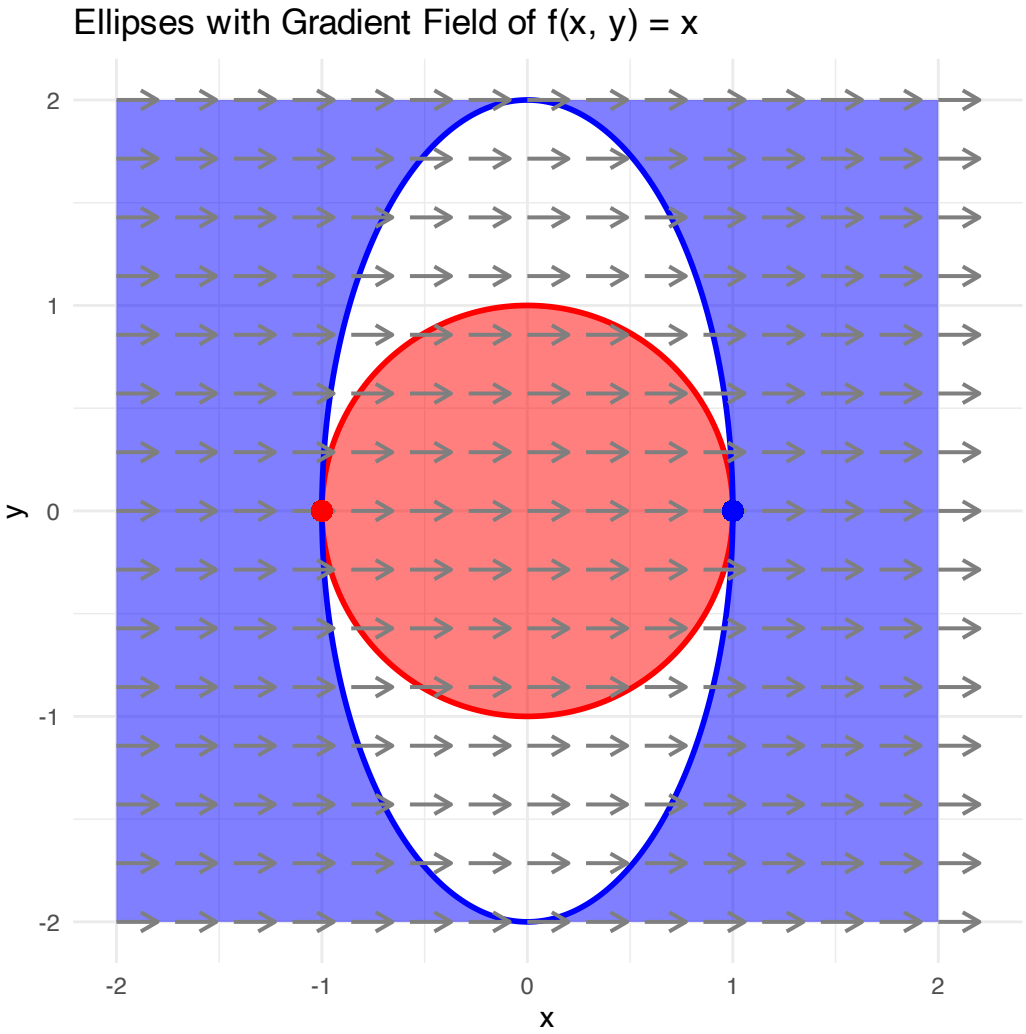


Figure 1.3: Feasible region and constrained minimum for the alternative objective.

1.2.9 Lagrangian Function

We define the Lagrangian function as follows:

$$\mathcal{L}(x, y, \mu_1, \mu_2) = x - \mu_1(x^2 + y^2 - 1) - \mu_2\left(1 - x^2 - \frac{1}{4}y^2\right)$$

1.2.10 First-Order Conditions

The first-order conditions derived from the Lagrangian are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 1 + 2x(\mu_2 - \mu_1), \\
\frac{\partial \mathcal{L}}{\partial y} &= y\left(\frac{1}{2}\mu_2 - 2\mu_1\right), \\
0 &= \mu_1(x^2 + y^2 - 1), \\
0 &= \mu_2\left(1 - x^2 - \frac{1}{4}y^2\right).
\end{aligned}$$

1.2.11 Solutions of First-Order Conditions

- **Case** $y = 0$

$$\begin{aligned}
0 &= 1 + 2x(\mu_2 - \mu_1), \\
0 &= \mu_1(x^2 - 1), \\
0 &= \mu_2(1 - x^2).
\end{aligned}$$

  in this case if $\mu_1 = \mu_2 = 0$ the first equation fails. Thus $x = 1$ or $x = -1$
  - **Case** $x = 1$ solving $1 + 2(\mu_2 - \mu_1) = 0$ we have $\mu_1 = 1/2 + \mu_2$
  - **Case** $x = -1$ solving $1 - 2(\mu_2 - \mu_1) = 0$ we have $\mu_2 = 1/2 + \mu_1$

  thus we have two family of solution
  - $x = 1$, $y = 0$, $\mu_1 = 1/2 + \mu_2$ with $\mu_1 \geq 0$
  - $x = -1$, $y = 0$, $\mu_2 = 1/2 + \mu_1$ with $\mu_2 \geq 0$
- **Case** $y \neq 0$ in this case $\mu_2 = 4\mu_1$ and

$$\begin{cases}
0 = 1 + 6x\mu_1, \\
0 = \mu_1(x^2 + y^2 - 1), \\
0 = 4\mu_1\left(1 - x^2 - \frac{1}{4}y^2\right).
\end{cases} \implies \mu_1 \neq 0 \implies \begin{cases}
0 = x^2 + y^2 - 1, \\
0 = 1 - x^2 - \frac{1}{4}y^2.
\end{cases}$$

  but the last system has solution $y = 0$ and $x = \pm 1$. Thus the solution must be discarded.

The solutions to the first-order conditions yield the following critical points:

$$\begin{aligned}
x = 1, \quad y = 0, \quad \mu_1 &= \frac{1}{2} + \mu_2, \\
x = -1, \quad y = 0, \quad \mu_2 &= \frac{1}{2} + \mu_1
\end{aligned}$$

1.2.12 Analysis of the candidate $(1, 0)$

Both inequalities are active:

$$g_1(1, 0) = g_2(1, 0) = 0.$$

Their gradients are

$$\nabla g_1(1, 0) = (2, 0)^\top, \qquad \nabla g_2(1, 0) = (-2, 0)^\top.$$

Thus LICQ fails. In fact MFCQ fails as well: there is no direction d such that simultaneously

$$\nabla g_1(1,0)^\top d > 0, \qquad \nabla g_2(1,0)^\top d > 0,$$

because these two inequalities require $d_1 > 0$ and $d_1 < 0$ at the same time. The KKT multipliers are therefore nonunique:

$$\mu_1 = \frac{1}{2} + \mu_2, \qquad \mu_2 \geq 0.$$

Because the standard constraint qualifications fail, it is safer not to infer minimality from a reduced-Hessian test alone. A direct feasible curve settles the question. For small t define

$$x(t) = \sqrt{1 - t^2}, \qquad y(t) = t.$$

Then $g_1(x(t), y(t)) = 0$ and

$$g_2(x(t), y(t)) = 1 - (1 - t^2) - \frac{1}{4}t^2 = \frac{3}{4}t^2 \geq 0,$$

so the curve is feasible. Moreover,

$$f(x(t), y(t)) = x(t) = \sqrt{1 - t^2} < 1 = f(1,0) \qquad (t \neq 0).$$

Hence $(1,0)$ is **not a local minimum** (indeed it is the local maximum of x on the feasible lens near that point).

1.2.13 Analysis of the candidate $(-1,0)$

Again both inequalities are active and

$$\nabla g_1(-1,0) = (-2,0)^\top, \qquad \nabla g_2(-1,0) = (2,0)^\top,$$

so LICQ and MFCQ fail. The KKT multipliers are nonunique:

$$\mu_2 = \frac{1}{2} + \mu_1, \qquad \mu_1 \geq 0.$$

Nevertheless, minimality is immediate from the second inequality alone. Every feasible point satisfies

$$x^2 + \frac{1}{4}y^2 \leq 1,$$

and therefore $x \geq -1$. Equality $x = -1$ forces $y = 0$. Consequently

$$f(x,y) = x \geq -1 = f(-1,0)$$

for every feasible point, with equality only at $(-1,0)$. Thus $(-1,0)$ is in fact the **unique global minimum**.

This example is useful precisely because the usual LICQ-based second-order KKT test is not applicable at the two intersection points. The geometry of the feasible set (or a more general second-order theory for degenerate constraints) must be used instead.

Example 5

Consider the problem

$$\begin{aligned} \text{minimize:} \quad & f(x,y) = 6 + 2y - x^2 \\ \text{equality:} \quad & h(x,y) = y - x^2 \end{aligned}$$

1.2.14 Lagrangian function

$$\mathcal{L}(x, y, \lambda) = 6 + 2y - x^2 - \lambda(y - x^2)$$

1.2.15 First-Order Conditions

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= 2x(\lambda - 1) = 0\\ \frac{\partial \mathcal{L}}{\partial y} &= 2 - \lambda = 0\\ 0 &= y - x^2\end{aligned}$$

1.2.16 Solution First-Order Conditions

- $\lambda = 2$
- $x = 0$
- $y = 0$

1.2.17 Analysis of the Candidate

Gradient of the constraints

$$\nabla(y - x^2) = \begin{pmatrix}-2x & 1\end{pmatrix}, \quad \Longrightarrow \quad \begin{pmatrix}0 & 1\end{pmatrix}$$

1.2.17.1 Check Qualification

The gradient $(0, 1)$ is nonzero, hence LICQ holds.

1.2.17.2 Kernel of the Gradient of Active Constraints

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} = z_2 = 0, \quad \Longrightarrow \quad \begin{pmatrix}z_1\\ z_2\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\end{pmatrix} = \underbrace{\begin{pmatrix}1\\ 0\end{pmatrix}}_{K^a}\alpha$$

1.2.17.3 Reduced Hessian

$$\nabla^2_{(x,y)}\mathcal{L}(x, y, \lambda) = \begin{pmatrix}2(\lambda - 1) & 0\\ 0 & 0\end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$$

$$K^{a,\top}\nabla^2_{(x,y)}\mathcal{L}(0, 0, 2)K^a = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (2)$$

The reduced Hessian is positive definite. Therefore the second-order sufficient condition holds and $(0, 0)$ is a strict local minimum. In fact, substituting $y = x^2$ into the objective gives $f = 6 + x^2$, so this minimum is global.

1.2.18 Why the curvature of the constraint matters

A useful companion test compares two constraints having exactly the same tangent space at the candidate. Consider

$$f(x, y) = 3 + y - \frac{1}{2}x^2$$

with either

$$h_c(x, y) = y - x^2 = 0 \qquad \text{or} \qquad h_s(x, y) = y = 0.$$

At $(0, 0)$,

$$\nabla h_c(0, 0) = \nabla h_s(0, 0) = (0, 1)^\top,$$

so the two feasible curves have the same tangent line. The first-order Lagrange conditions also give the same multiplier, $\lambda = -1$, if the convention $\mathcal{L} = f + \lambda h$ is used. Nevertheless,

$$\nabla^2 f(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

while

$$\nabla^2_{(x,y)} \mathcal{L}_c(0,0,-1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \nabla^2_{(x,y)} \mathcal{L}_s(0,0,-1) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The common tangent space is spanned by $z = (1,0)^\top$. Hence

$$z^\top \nabla^2 \mathcal{L}_c z = 1 > 0, \qquad z^\top \nabla^2 \mathcal{L}_s z = -1 < 0.$$

Thus $(0,0)$ is a strict local minimum on the parabola $y = x^2$, but a strict local maximum on the line $y = 0$. This small example is important: **for curved constraints, the correct second-order object is the Hessian of the Lagrangian, not the Hessian of the objective alone**. This is the standard second-order KKT principle; see Nocedal and Wright, *Numerical Optimization*, 2nd ed., Section 12.4.

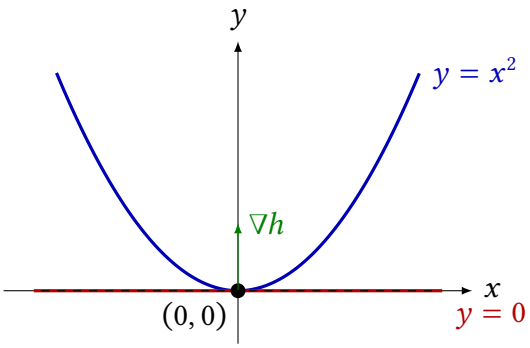


Figure 1.4: A straight and a curved constraint with the same tangent and the same normal at the origin. Their different curvature changes the reduced Hessian of the Lagrangian.

1.3 Example 6: A Three-Variable KKT Test with a Weakly Active Constraint

Consider

$$\begin{aligned} \min_{x,y,z} \quad & f(x,y,z) = -x^3 + y^2 - 2xz^2, \\ \text{subject to} \quad & h(x,y,z) = 2x + y^2 + z - 5 = 0, \\ & g_1(x,y,z) = 5x^2 - y^2 - z - 2 \geq 0, \\ & g_2(x,y,z) = x \geq 0, \qquad g_3(x,y,z) = y \geq 0, \qquad g_4(x,y,z) = z \geq 0. \end{aligned}$$

Using the convention

$$\mathcal{L} = f - \lambda h - \sum_{j=1}^{4} \mu_j g_j, \qquad \mu_j \geq 0,$$

the KKT equations admit

$$x^\star = (1,0,3)^\top, \qquad \lambda^\star = -\frac{47}{4}, \qquad \mu_1^\star = \frac{1}{4}, \qquad \mu_2^\star = \mu_3^\star = \mu_4^\star = 0.$$

At this point h, g_1, and g_3 are active. Their gradients are

$$\nabla h = (2,0,1)^\top, \qquad \nabla g_1 = (10,0,-1)^\top, \qquad \nabla g_3 = (0,1,0)^\top.$$

They are linearly independent, so LICQ holds.

1.3.1 The critical cone, not merely the null space of all active gradients

There is a subtle point in this exercise. Since $\mu_3^\star = 0$, the active constraint $y \geq 0$ is **weakly active**. For second-order conditions we should therefore use the critical cone. The equalities associated with h and with the strictly positive multiplier $\mu_1^\star$ give

$$2d_x + d_z = 0, \qquad 10d_x - d_z = 0,$$

hence $d_x = d_z = 0$. The weakly active bound gives $d_y \geq 0$. Thus

$$\mathcal{C}(\boldsymbol{x}^\star, \boldsymbol{\mu}^\star) = \{(0, \eta, 0)^\top \,:\, \eta \geq 0\}.$$

The Hessian of the Lagrangian at the KKT point is

$$\nabla^2_{\boldsymbol{xx}} \mathcal{L}(\boldsymbol{x}^\star, \lambda^\star, \boldsymbol{\mu}^\star) = \begin{pmatrix} -\frac{17}{2} & 0 & -12 \\ 0 & 26 & 0 \\ -12 & 0 & -4 \end{pmatrix}.$$

For every nonzero critical direction $\boldsymbol{d} = (0, \eta, 0)^\top$,

$$\boldsymbol{d}^\top \nabla^2 \mathcal{L}\, \boldsymbol{d} = 26\eta^2 > 0.$$

Hence the second-order sufficient condition holds and $\boldsymbol{x}^\star$ is a strict local minimizer. This example is particularly useful for distinguishing **active** constraints from **strongly active** constraints and for motivating the critical cone used in the general second-order KKT theory.

1.4 Example 7: Chong–Zak Maximum-Power-Transfer Test

This is a standard textbook example; it appears as Example 20.2 in Chong and Zak, *An Introduction to Optimization*. A 20 V source feeds two series resistors, a fixed 10 Ω resistor and a variable resistor $R \geq 0$. Let i be the circuit current. The power absorbed by R is $P = Ri^2$, so the maximization can be written as

$$\begin{aligned} \min_{R,i} \quad & -Ri^2, \\ \text{subject to} \quad & 20 - (R + 10)i = 0, \\ & R \geq 0. \end{aligned}$$

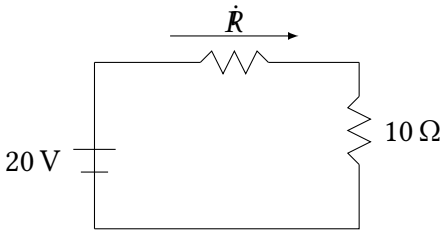


Figure 1.5: Circuit used in the Chong–Zak maximum-power-transfer example.

With

$$\mathcal{L} = -Ri^2 - \lambda[20 - (R + 10)i] - \mu R,$$

the interior KKT solution is

$$R^\star = 10, \qquad i^\star = 1, \qquad \lambda^\star = 1, \qquad \mu^\star = 0.$$

The bound is inactive. The equality-gradient row is

$$\nabla h(R^\star, i^\star) = (-1, -20),$$

so a tangent direction is $z = (-20, 1)^\top$. Moreover,

$$\nabla^2_{(R,i)}\mathcal{L}(R^\star, i^\star) = \begin{pmatrix} 0 & -1 \\ -1 & -20 \end{pmatrix},$$

and

$$z^\top \nabla^2 \mathcal{L} z = 20 > 0.$$

Thus the point is a strict local minimum of $-Ri^2$, equivalently a strict local maximum of the absorbed power. Eliminating the equality gives

$$P(R) = \frac{400R}{(R+10)^2},$$

whose unique maximizer on $R \geq 0$ is $R = 10\,\Omega$. This is the familiar **maximum power transfer theorem** for a purely resistive source/load network.

1.5 Example 8: Maximum-Volume Rectangular Box with Fixed Surface Area

A classical geometric optimization problem is to maximize the volume of a rectangular box with fixed surface area. With the normalization used here,

$$\begin{aligned} \min_{x,y,z} \quad & -xyz, \\ \text{subject to} \quad & 2(xy + yz + xz) - 1 = 0, \\ & x \geq 0, \qquad y \geq 0, \qquad z \geq 0. \end{aligned}$$

At an interior solution the bound multipliers vanish. By symmetry, or directly from the KKT equations,

$$x^\star = y^\star = z^\star = \frac{1}{\sqrt{6}}, \qquad \lambda^\star = -\frac{\sqrt{6}}{24}.$$

(The all-negative stationary point of the equality-constrained equations is not feasible and must be discarded once the nonnegativity constraints are included.)

A basis for the tangent space of the equality constraint is, for example,

$$Z = \begin{pmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The reduced Hessian is

$$Z^\top \nabla^2 \mathcal{L}(x^\star, \lambda^\star) Z = \begin{pmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{12} \\ \frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{6} \end{pmatrix},$$

whose leading principal minors are positive. Hence the second-order sufficient condition holds.

There is also a short global argument. Since

$$xy + yz + xz = \frac{1}{2},$$

AM–GM gives

$$\frac{xy + yz + xz}{3} \geq \sqrt[3]{(xy)(yz)(xz)} = (xyz)^{2/3},$$

and therefore

$$xyz \leq \frac{1}{6\sqrt{6}},$$

with equality only for $xy = yz = xz$, hence $x = y = z = 1/\sqrt{6}$. Thus the KKT candidate is the unique global maximizer of the volume.

1.6 Example 9: Nonconvex Objective on an Elliptic Feasible Set

Consider the course test

$$\begin{aligned} \min_{x,y} \quad & f(x,y) = x^2\left(1 + \frac{1}{10}\sin(1-10x)\right) + y^2 - e^y, \\ \text{subject to} \quad & g(x,y) = 1 - x^2 - 2y^2 \geq 0. \end{aligned}$$

The feasible set is the ellipse $x^2 + 2y^2 \leq 1$. Numerical solution of the KKT system gives the boundary candidate

$$x^\star = 0, \qquad y^\star = \frac{1}{\sqrt{2}}, \qquad \mu^\star \approx 0.2170469283.$$

For the convention $\mathcal{L} = f - \mu g$, the tangent space to the active constraint is spanned by $z = (1,0)^\top$. The important detail is that the second-order test must use the Hessian of the **Lagrangian**:

$$\nabla^2\mathcal{L} = \nabla^2 f - \mu\nabla^2 g.$$

At the candidate,

$$z^\top\nabla^2\mathcal{L}z = f_{xx}(0, 1/\sqrt{2}) + 2\mu^\star \approx 2.6024 > 0.$$

Thus the second-order sufficient condition is satisfied on the one-dimensional critical/tangent space, and the candidate is a strict local minimizer. This example is not a widely standardized named benchmark; it is useful as a compact numerical KKT test with a nonconvex objective and an active curved inequality.

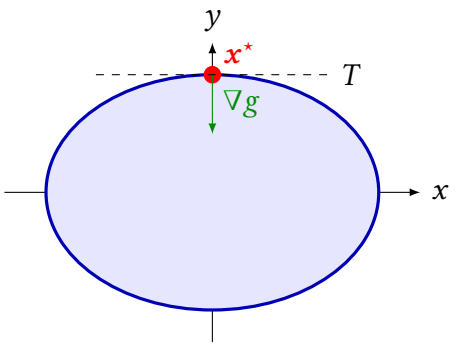


Figure 1.6: Elliptic feasible set and tangent line at the KKT point of the nonconvex test problem.

1.7 Example 10: A Linear Objective on the Unit Sphere

Consider

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^3} \quad & f(\boldsymbol{x}) = x + y + z, \\ \text{subject to} \quad & h(\boldsymbol{x}) = x^2 + y^2 + z^2 - 1 = 0. \end{aligned}$$

This is a useful first equality-constrained problem because the geometry is completely transparent and the same answer can be obtained by parameterization, by Cauchy–Schwarz, or by Lagrange multipliers. The original exercise develops both a spherical parameterization

and the multiplier approach; here we emphasize the latter and use the parameterization only as a geometric check.

The Lagrangian is

$$\mathcal{L}(\boldsymbol{x}, \lambda) = x + y + z - \lambda(x^2 + y^2 + z^2 - 1).$$

The first-order conditions are

$$\nabla_{\boldsymbol{x}} \mathcal{L} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}, \qquad x^2 + y^2 + z^2 = 1.$$

Hence

$$x = y = z = \frac{1}{2\lambda},$$

and therefore

$$3\frac{1}{4\lambda^2} = 1, \qquad \lambda = \pm\frac{\sqrt{3}}{2}.$$

The two stationary points are

$$\boldsymbol{x}_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_- = -\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Their objective values are

$$f(\boldsymbol{x}_+) = \sqrt{3}, \qquad f(\boldsymbol{x}_-) = -\sqrt{3}.$$

Thus $\boldsymbol{x}_-$ is the global minimum and $\boldsymbol{x}_+$ is the global maximum. This also follows immediately from Cauchy–Schwarz:

$$|x + y + z| \le \sqrt{1^2 + 1^2 + 1^2}\sqrt{x^2 + y^2 + z^2} = \sqrt{3}.$$

At either point the tangent space is

$$T(\boldsymbol{x}_\pm) = \left\{ \boldsymbol{d} \,:\, \boldsymbol{x}_\pm^{\mathsf{T}} \boldsymbol{d} = 0 \right\}.$$

Since

$$\nabla^2_{\boldsymbol{x}\boldsymbol{x}} \mathcal{L} = -2\lambda \boldsymbol{I},$$

the reduced Hessian is positive definite at $\boldsymbol{x}_-$ and negative definite at $\boldsymbol{x}_+$.

The main lesson is that the multiplier condition is simply the normality condition

$$\nabla f(\boldsymbol{x}^\star) \in \operatorname{span}\{\nabla h(\boldsymbol{x}^\star)\}.$$

1.8 Example 11: Two Equality Constraints in $\mathbb{R}^3$

Consider

$$\begin{aligned} \min_{x,y,z} \quad & f(x, y, z) = x^2 - z, \\ \text{subject to} \quad & h_1(x, y, z) = y - x^2 = 0, \\ & h_2(x, y, z) = x + y + z - 1 = 0. \end{aligned}$$

This problem is especially useful because it can first be reduced to one variable by eliminating the constraints, and then solved again with Lagrange multipliers and a reduced Hessian.

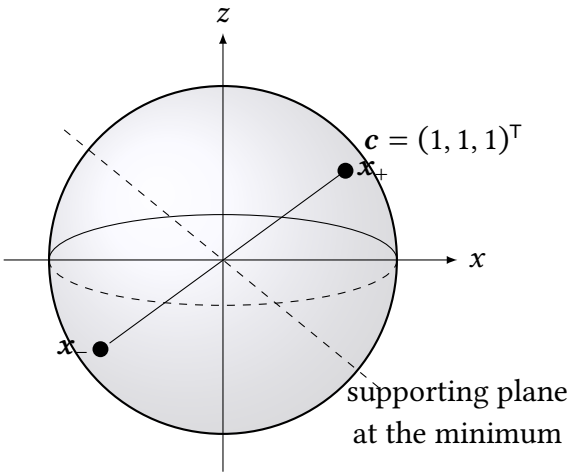


Figure 1.7: Geometric interpretation of a linear objective on the unit sphere. The extrema lie where the sphere normal is parallel to $\boldsymbol{c} = (1, 1, 1)^{\mathsf{T}}$.

1.8.1 Direct elimination

From the constraints,

$$y = x^2, \qquad z = 1 - x - x^2.$$

Therefore

$$\widehat{f}(x) = x^2 - (1 - x - x^2) = 2x^2 + x - 1.$$

Hence

$$\widehat{f}'(x) = 4x + 1 = 0, \qquad x^\star = -\frac{1}{4},$$

and

$$\widehat{f}''(x) = 4 > 0.$$

Thus

$$\boxed{\boldsymbol{x}^\star = \begin{bmatrix} -1/4 \\ 1/16 \\ 19/16 \end{bmatrix}}$$

is a strict minimum along the feasible curve.

1.8.2 Lagrange multiplier calculation

Using

$$\mathcal{L} = f - \lambda_1 h_1 - \lambda_2 h_2,$$

stationarity gives

$$\begin{aligned} 2x + 2\lambda_1 x - \lambda_2 &= 0, \\ -\lambda_1 - \lambda_2 &= 0, \\ -1 - \lambda_2 &= 0. \end{aligned}$$

Therefore

$$\lambda_2 = -1, \qquad \lambda_1 = 1, \qquad x = -\frac{1}{4},$$

which reproduces the point obtained by elimination.

The constraint Jacobian at $x^\star$ is

$$J_h(x^\star) = \begin{bmatrix} 1/2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

It has rank two, so LICQ holds. A basis of the one-dimensional tangent space is

$$Z = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \qquad J_h Z = 0.$$

With the sign convention above,

$$\nabla^2_{xx}\mathcal{L}(x^\star) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the reduced Hessian is the scalar

$$Z^\top \nabla^2_{xx}\mathcal{L} Z = 16 > 0.$$

Therefore the second-order sufficient condition is satisfied.

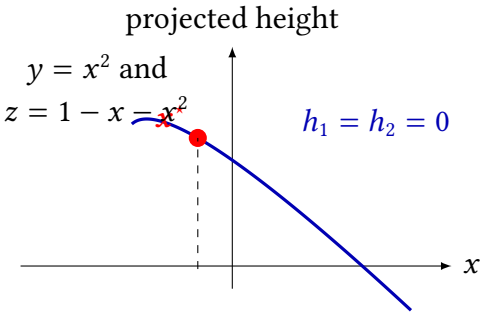


Figure 1.8: A schematic projection of the one-dimensional feasible curve obtained by intersecting the two equality constraints. The constrained problem reduces locally to a one-variable problem.

This example makes the dimension count explicit:

$$\dim T = n - m = 3 - 2 = 1.$$

1.9 Example 12: Positive Curvature Only on the Constraint Kernel

Second-order constrained optimality does **not** require the Hessian to be positive definite in the entire ambient space. It only needs positive curvature along feasible first-order directions.

The historical worksheet motivating this example uses

$$A = \begin{bmatrix} 4 & 1 & 1 & -4 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 4 & -1 \\ -4 & -1 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

There is an important technical point: the displayed A is not symmetric, so it is not meaningful to call A itself SPD. For a quadratic form only the symmetric part matters. Define

$$S = \frac{1}{2}(A + A^\top) = \begin{bmatrix} 4 & 1 & 1 & -4 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ -4 & 0 & 0 & 2 \end{bmatrix}.$$

The kernel of B can be written as

$$d = \begin{bmatrix} -y + z \\ y \\ z \\ -z \end{bmatrix} = Z \begin{bmatrix} z \\ y \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

The symmetric reduced matrix is

$$Z^\top S Z = \begin{bmatrix} 20 & -7 \\ -7 & 4 \end{bmatrix}.$$

By Sylvester's criterion,

$$20 > 0, \qquad \det \begin{bmatrix} 20 & -7 \\ -7 & 4 \end{bmatrix} = 31 > 0,$$

hence

$$Z^\top S Z \succ 0.$$

Equivalently, for every nonzero $d \in \ker B$,

$$d^\top S d = 4y^2 - 14yz + 20z^2 > 0.$$

Thus the quadratic form has positive curvature on the feasible tangent subspace even though positivity in the full ambient space is not required.

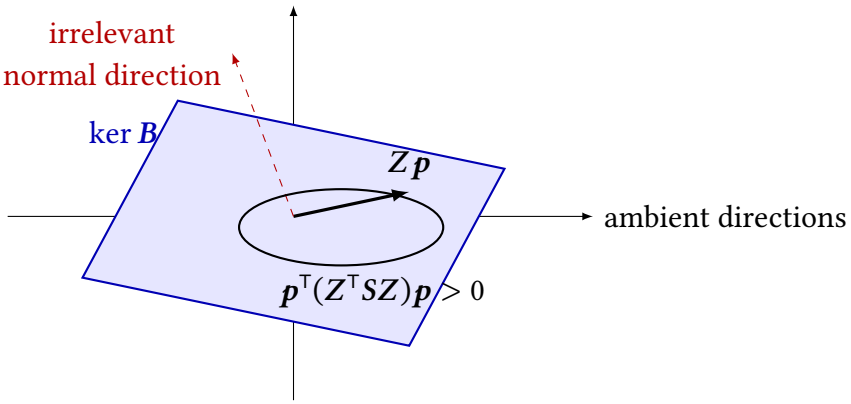


Figure 1.9: Reduced-Hessian principle: only curvature along directions in $\ker B$ is relevant to the equality-constrained second-order test.

This is the algebraic core of the reduced-Hessian test used throughout the KKT examples.

1.10 Example 13: Box Constraints, Active Sets, and KKT Filtering

Consider the box-constrained problem

$$\begin{aligned} \min_{x,y} \quad & f(x, y) = y - x^2 + x, \\ \text{subject to} \quad & -1 \le x \le 1, \\ & -1 \le y \le 1. \end{aligned}$$

Write the inequalities in the course convention $g_i \geq 0$:

$$g_1 = 1 - x, \quad g_2 = x + 1,$$
$$g_3 = 1 - y, \quad g_4 = y + 1.$$

The Lagrangian is

$$\mathcal{L} = f - \sum_{i=1}^{4} \mu_i g_i, \qquad \mu_i \geq 0.$$

Stationarity gives

$$1 - 2x + \mu_1 - \mu_2 = 0,$$
$$1 + \mu_3 - \mu_4 = 0,$$

together with

$$\mu_i g_i = 0, \qquad g_i \geq 0, \qquad \mu_i \geq 0.$$

There is no unconstrained stationary point because

$$\frac{\partial f}{\partial y} = 1.$$

Consequently every constrained minimizer must lie on the boundary. The KKT system produces three feasible dual-feasible candidates.

1.10.1 Candidate A: $(-1, -1)$

The active constraints are $g_2 = g_4 = 0$. Stationarity yields

$$\mu_2 = 3, \qquad \mu_4 = 1,$$

with the remaining multipliers zero. Since both active multipliers are strictly positive, the critical cone is reduced to $\{\mathbf{0}\}$ at this corner. Direct evaluation confirms

$$f(-1, -1) = -3.$$

1.10.2 Candidate B: $(1, -1)$

Here $g_1 = g_4 = 0$ and

$$\mu_1 = 1, \qquad \mu_4 = 1.$$

Again the critical cone at the vertex is trivial. The objective value is

$$f(1, -1) = -1.$$

This point is a local minimum at the corner, but not the global one.

1.10.3 Candidate C: $(1/2, -1)$

Only g_4 is active, with

$$\mu_4 = 1.$$

The feasible tangent direction is horizontal,

$$\boldsymbol{d} = \begin{bmatrix} d_x \\ 0 \end{bmatrix}.$$

Since

$$\nabla^2 f = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},$$

we obtain

$$\boldsymbol{d}^{\mathsf{T}}\nabla^2 f\,\boldsymbol{d} = -2d_x^2 < 0 \qquad (d_x \neq 0).$$

Thus this KKT point is **not** a local minimum; it is a saddle point relative to the box.

The remaining boundary stationary points that appear if one enumerates faces without enforcing multiplier signs are rejected by dual feasibility. In particular, the maximizer

$$\left(\frac{1}{2}, 1\right)$$

is not a KKT point for the minimization problem with the sign convention $g_i \geq 0$, $\mathcal{L} = f - \sum_i \mu_i g_i$, $\mu_i \geq 0$.

Finally, because f is increasing in y, the global minimizer must have $y = -1$. On that edge,

$$f(x, -1) = -1 - x^2 + x,$$

which is concave, so its minimum over $[-1, 1]$ occurs at an endpoint. Comparing -3 and -1 gives

$$\boxed{(x^\star, y^\star) = (-1, -1)}, \qquad \boxed{f^\star = -3}.$$

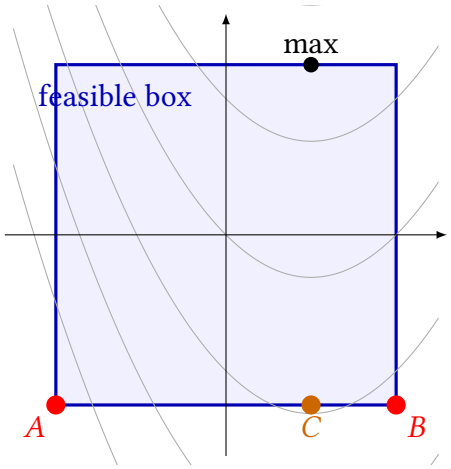


Figure 1.10: Box-constrained test. Gray curves are level sets of $f(x, y) = y - x^2 + x$. The KKT minimization candidates are $A = (-1, -1)$, $B = (1, -1)$, and $C = (1/2, -1)$; the reduced-Hessian test rejects C.

1.11 Example 14: A Mixed Equality–Inequality Problem with a Strict Local Minimum

Consider

$$\min_{x,y,z} \; f(x, y, z) = -xyz - z$$

subject to

$$h(x, y, z) = x + y - 1 = 0, \qquad g(x, y, z) = 2 - z \geq 0.$$

This example is useful because it contains both an equality and an inequality constraint, it has several KKT points, and the strict local minimizer is **not** a global minimizer.

With the convention $g \geq 0$, define

$$\mathcal{L}(x, y, z, \lambda, \mu) = -xyz - z - \lambda(x + y - 1) - \mu(2 - z), \qquad \mu \geq 0.$$

The KKT conditions are

$$\begin{aligned} -yz - \lambda &= 0, \\ -xz - \lambda &= 0, \\ -xy - 1 + \mu &= 0, \\ x + y - 1 &= 0, \\ \mu(2 - z) &= 0, \\ \mu \geq 0, \qquad & 2 - z \geq 0. \end{aligned}$$

Subtracting the first two stationarity equations gives

$$z(x - y) = 0.$$

1.11.1 Active inequality: $z = 2$

If $\mu > 0$, complementarity gives $z = 2$. Then the first two stationarity equations imply $x = y$, and the equality constraint gives

$$x^\star = y^\star = \frac{1}{2}, \qquad z^\star = 2.$$

The multipliers are

$$\lambda^\star = -1, \qquad \mu^\star = 1 + x^\star y^\star = \frac{5}{4} > 0.$$

Hence

$$\boldsymbol{x}^\star = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$$

satisfies the KKT conditions with the inequality strongly active.

The gradients of the active constraints are

$$\nabla h(\boldsymbol{x}^\star) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \nabla g(\boldsymbol{x}^\star) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

which are linearly independent. Therefore LICQ holds. The tangent space to the active constraints is

$$\mathcal{T} = \left\{ \boldsymbol{d} \,:\; d_x + d_y = 0,\; d_z = 0 \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Since both constraints are affine, the Hessian of the Lagrangian is just the Hessian of the objective,

$$\nabla^2_{xx}\mathcal{L} = \begin{bmatrix} 0 & -z & -y \\ -z & 0 & -x \\ -y & -x & 0 \end{bmatrix}.$$

At $\boldsymbol{x}^\star$,

$$\boldsymbol{H}^\star = \begin{bmatrix} 0 & -2 & -1/2 \\ -2 & 0 & -1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix}.$$

For $\boldsymbol{d} = \alpha(1, -1, 0)^T$,

$$\boldsymbol{d}^T \boldsymbol{H}^\star \boldsymbol{d} = 4\alpha^2 > 0 \qquad (\alpha \neq 0).$$

Therefore $\boldsymbol{x}^\star$ is a **strict local minimizer**.

1.11.2 Inactive inequality: two additional KKT points

If $\mu = 0$, stationarity in z gives $xy = -1$. Together with $x + y = 1$,

$$x = \frac{1 \pm \sqrt{5}}{2}, \qquad y = \frac{1 \mp \sqrt{5}}{2}.$$

The first two stationarity equations then force $z = 0$ and $\lambda = 0$. Thus there are two additional KKT points,

$$\left(\frac{1 \pm \sqrt{5}}{2}, \frac{1 \mp \sqrt{5}}{2}, 0 \right),$$

but they are saddle points. Indeed, after eliminating $y = 1 - x$ the objective becomes

$$\phi(x, z) = -z \left(1 + x - x^2\right),$$

and at either of the above points its Hessian has eigenvalues of opposite sign.

Finally, the problem is not globally bounded below. For instance, with $z = -1$ and $y = 1 - x$,

$$f(x, 1 - x, -1) = 1 + x - x^2 \longrightarrow -\infty \qquad \text{as } |x| \to \infty.$$

Thus this example also illustrates the distinction between a strict **local** minimizer and a global minimizer.

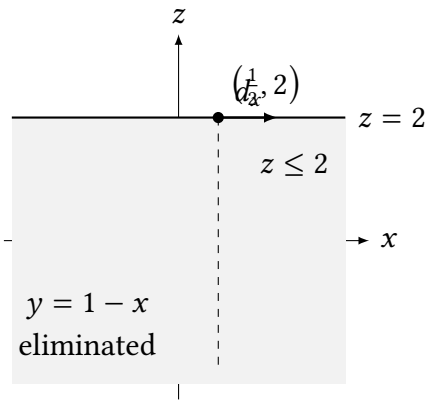


Figure 1.11: Reduced (x, z) view after eliminating $y = 1 - x$. The active inequality is the line $z = 2$; the strict local minimizer lies on this boundary.

1.12 Example 15: KKT Filtering with an Elliptic Constraint and Box Constraint in z

Consider

$$\min_{x,y,z} \; f(x, y, z) = x + y^2 - z$$

subject to

$$g_1(x, y, z) = 1 - \frac{x^2}{2} - y^2 \geq 0, \qquad g_2(x, y, z) = z \geq 0, \qquad g_3(x, y, z) = 1 - z \geq 0.$$

The feasible set is an elliptic cylinder truncated by the planes $z = 0$ and $z = 1$.
The Lagrangian is

$$\mathcal{L} = x + y^2 - z - \mu_1 \left(1 - \frac{x^2}{2} - y^2\right) - \mu_2 z - \mu_3(1 - z),$$

with $\mu_i \geq 0$. Stationarity gives

$$\begin{aligned} 1 + \mu_1 x &= 0, \\ 2y(1 + \mu_1) &= 0, \\ -1 - \mu_2 + \mu_3 &= 0. \end{aligned}$$

Together with

$$\mu_1 g_1 = 0, \qquad \mu_2 z = 0, \qquad \mu_3(1 - z) = 0,$$

these equations allow a short active-set analysis.
Because $1 + \mu_1 x = 0$, we must have $\mu_1 > 0$. Hence $g_1 = 0$. Since $\mu_1 \geq 0$,

$$2y(1 + \mu_1) = 0 \quad \Longrightarrow \quad y = 0.$$

Thus

$$1 - \frac{x^2}{2} = 0,$$

and stationarity in x selects the negative root,

$$x^\star = -\sqrt{2}, \qquad \mu_1^\star = \frac{1}{\sqrt{2}}.$$

For the z constraints, an interior value $0 < z < 1$ would imply $\mu_2 = \mu_3 = 0$, contradicting the third stationarity equation. At $z = 0$, one would obtain $\mu_2 = -1$, which violates dual feasibility. Therefore

$$z^\star = 1, \qquad \mu_2^\star = 0, \qquad \mu_3^\star = 1.$$

The unique KKT point is therefore

$$\boxed{\boldsymbol{x}^\star = (-\sqrt{2}, 0, 1)^T}$$

with

$$\boxed{\boldsymbol{\mu}^\star = \left(\frac{1}{\sqrt{2}}, 0, 1\right)^T}.$$

The active constraints are g_1 and g_3. Their gradients are

$$\nabla g_1(\boldsymbol{x}^\star) = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \qquad \nabla g_3(\boldsymbol{x}^\star) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

so LICQ holds and

$$\mathcal{T} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The Hessian of the Lagrangian is

$$\nabla^2_{xx}\mathcal{L} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & 2 + 2\mu_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

At the KKT point, for $\boldsymbol{d} = \alpha(0, 1, 0)^T$,

$$\boldsymbol{d}^T \nabla^2_{xx} \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\mu}^\star)\boldsymbol{d} = (2+\sqrt{2})\alpha^2 > 0 \qquad (\alpha \neq 0).$$

Hence the second-order sufficient condition holds and $\boldsymbol{x}^\star$ is a strict local minimizer.

In this case one can also prove global optimality directly. Since $-z$ is minimized by $z = 1$, it remains to minimize

$$x + y^2$$

over $x^2/2 + y^2 \leq 1$. For fixed y the smallest admissible x is

$$x = -\sqrt{2(1-y^2)},$$

so the reduced problem is

$$\min_{|y|\leq 1} \left[-\sqrt{2(1-y^2)} + y^2 \right],$$

whose unique minimizer is $y = 0$. Therefore $\boldsymbol{x}^\star$ is the unique global minimizer.

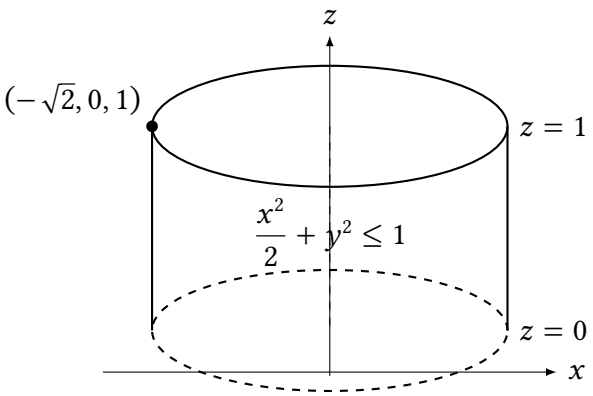


Figure 1.12: Schematic geometry of the feasible elliptic cylinder. The objective drives z to its upper bound and x to the leftmost point of the ellipse.

1.13 Example 16: A Continuum of Global Minimizers and Failure of Strict Second-Order Sufficiency

Consider

$$\min_{x,y} \; f(x,y) = x^2$$

subject to

$$g_1(x,y) = 1 - x^2 - y^2 \geq 0, \qquad g_2(x,y) = x + y \geq 0.$$

The objective satisfies $f(x,y) \geq 0$. Every feasible point with $x = 0$ therefore gives the smallest possible objective value. Feasibility requires

$$1 - y^2 \geq 0, \qquad y \geq 0,$$

hence the complete set of global minimizers is

$$\boxed{\mathcal{X}^\star = \{(0,y) \,:\, 0 \leq y \leq 1\}.}$$

Thus the minimizer is not isolated.

The Lagrangian is

$$\mathcal{L}(x,y,\mu_1,\mu_2) = x^2 - \mu_1(1 - x^2 - y^2) - \mu_2(x+y),$$

with $\mu_1, \mu_2 \geq 0$. Stationarity is

$$2x(1 + \mu_1) - \mu_2 = 0,$$
$$2\mu_1 y - \mu_2 = 0.$$

For every point $(0, y)$ with $0 \leq y \leq 1$, the choice

$$\mu_1^\star = \mu_2^\star = 0$$

satisfies stationarity, dual feasibility, and complementarity. Therefore the entire segment consists of KKT points.

For an interior point of the segment, $0 < y < 1$, neither constraint is active. The Hessian of the Lagrangian is simply

$$\nabla^2_{(x,y)} \mathcal{L} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\boldsymbol{d}^T \nabla^2 \mathcal{L}\, \boldsymbol{d} = 2d_x^2 \geq 0,$$

so the second-order **necessary** condition holds. However, for the nonzero direction

$$\boldsymbol{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we obtain

$$\boldsymbol{d}^T \nabla^2 \mathcal{L}\, \boldsymbol{d} = 0.$$

Therefore the usual strict second-order sufficient condition fails. This is exactly what should happen: there is a flat feasible direction along the complete segment of minimizers, so none of these points is a strict local minimizer.

At the endpoints $(0, 0)$ and $(0, 1)$ one inequality becomes active, but its multiplier is still zero. These are examples of **weakly active constraints**. The critical cone contains one-sided feasible directions along the minimizing segment, and the quadratic form still vanishes on those directions.

This example is useful for separating three notions:

1. KKT conditions may hold on a whole continuum of points;
2. positive semidefiniteness of the reduced Hessian is only a necessary condition;
3. strict positive definiteness is sufficient for an isolated strict minimum, but it is not necessary for a non-strict minimum.

This exercise illustrates the practical active-set workflow: identify active constraints, enforce complementarity and multiplier signs, and only then apply the appropriate second-order test on the critical cone.

References for the Additional Tests

- E. K. P. Chong and S. H. Zak, *An Introduction to Optimization*, Wiley, Chapter 20, Example 20.2 (maximum-power-transfer KKT example).
- J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd ed., Springer, 2006, Section 12.4 (second-order necessary and sufficient conditions, critical cones, and the Hessian of the Lagrangian).

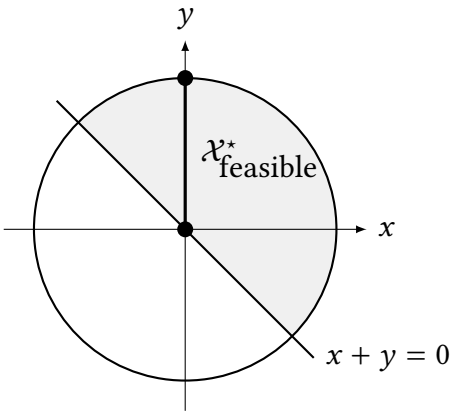


Figure 1.13: The feasible half disk and the complete segment of global minimizers $\mathcal{X}^\star = \{(0, y) : 0 \leq y \leq 1\}$.

The remaining added problems are course/exercise tests rather than standard named optimization benchmarks; they are included because they isolate useful KKT phenomena: curvature of the constraint, weak activity, reduced Hessians, and nonconvexity.

II DAE

2 Case Studies 43

2.1 Example 1: Nonlinear Reduction
2.2 Example 2: 3D Pendulum
2.3 Example 4: Double Pendulum
2.4 Example 7: Circuit DAE
2.5 Example 8: Index-5 Crane

3 DAE Exercises 83

3.1 Fundamental
3.2 Computational
3.3 Advanced

Case studies and exercises on differential-algebraic equations. The corresponding theory is in the part of the same name of the course notes.

2. Case Studies

The five models below combine several of the techniques of this part and are therefore collected separately rather than attached to a single chapter. Each is self-contained and can be assigned or presented on its own.

The examples that illustrate one technique only stayed in the course notes, in the Examples section of the corresponding chapter: the high-index chain in General Index Reduction, the index-three nonlinear system in Semi-Explicit Form, the coupled pendulums in Dummy Variables, and the index-two Van der Pol oscillator in Numerical Methods for DAEs.

2.1 Example 1: Nonlinear Reduction

Consider the DAE

$$\begin{cases} x'(t) = u(t) \\ y'(t) = v(t) \\ 2u'(t) + v'(t) + 2\,t\,x(t)\,\lambda(t) = t \\ u'(t) + 2\,v'(t) - \lambda(t) = 1 - t \\ t\,x(t)^2 - y(t) = 1 \end{cases}$$

define

$$Z = \begin{pmatrix} x \\ y \\ u \\ v \\ \lambda \end{pmatrix}, \quad E(Z,t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G(Z,t) = \begin{pmatrix} u \\ v \\ t - 2\,t\,x\,\lambda \\ 1 - t + \lambda \\ 1 + y - t\,x^2 \end{pmatrix}$$

so that the DAE can be written as

$$E(Z,t)\,Z'(t) = G(Z,t)$$

2.1.1 Index Reduction

The table

$$T = \left[\begin{array}{c|c} E(Z,t) & G(Z,t) \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 2 & 1 & 0 & t - 2\,t\,x\,\lambda \\ 0 & 0 & 1 & 2 & 0 & 1 - t + \lambda \\ 0 & 0 & 0 & 0 & 0 & 1 + y - t\,x^2 \end{array}\right]$$

is used to memorize the operation done on the DAE. The reduction is done by using linear algebra and avoid explicit construction of LU decomposition.

2.1.1.1 First Reduction

1. Exchange row 3 and 4 by using

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$T_1 = P_1 T = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 2 & 0 & 1 - t + \lambda \\ 0 & 0 & 2 & 1 & 0 & t - 2\,t\,x\,\lambda \\ 0 & 0 & 0 & 0 & 0 & 1 + y - t\,x^2 \end{array}\right]$$

2. Subtract 2 times row 3 to row 4

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$T_2 = L_2 T_1 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 2 & 0 & 1 - t + \lambda \\ 0 & 0 & 0 & -3 & 0 & 3t - 2(\lambda(1 + t\,x) + 1) \\ \hline 0 & 0 & 0 & 0 & 0 & 1 + y - t\,x^2 \end{array}\right]$$

3. Scale the fourth row by $-1/3$.

$$T_2 = L_2 T_1 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 2 & 0 & 1 - t + \lambda \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 0 & 0 & 0 & 0 & 0 & 1 + y - t\,x^2 \end{array}\right]$$

Now algebraic constraint is separated and is

$$[\text{alg1}] = 1 + y - t\,x^2 = 0$$

$$\Longrightarrow \qquad [\text{alg2}] = \frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} = y' - x^2 - 2txx' = 0$$

apply derivation to the algebraic constraint (reduce the index by one)

$$T_3 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 2 & 0 & 1 - t + \lambda \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline -2tx & 1 & 0 & 0 & 0 & x^2 \end{array}\right]$$

2.1.1.2 Second Reduction

1. Eliminate the entries $-2tx$ and 1 from the last row by multiplying

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2tx & -1 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$T_4 = L_3 T_3 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 2 & 0 & 1 - t + \lambda \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 0 & 0 & 0 & 0 & 0 & x^2 + 2t\,x\,u - v \end{array}\right]$$

2. Eliminate the remaining entry 2 by multiplying

$$U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$T_5 = U_4 T_4 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & t - \frac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 0 & 0 & 0 & 0 & 0 & x^2 + 2t\,x\,u - v \end{array}\right]$$

The algebraic constraint is

$$[\text{alg2}] = x^2 + 2t\,x\,u - v = 0$$

$$\Longrightarrow \qquad [\text{alg3}] = \frac{\mathrm{d}[\text{alg2}]}{\mathrm{d}t} = 2(x + tu)x' + 2x\,u + 2tx\,u' - v' = 0$$

apply derivation to the algebraic constraint (reduce the index by one)

$$T_6 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & t - \frac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 2(x + tu) & 0 & 2tx & -1 & 0 & -2x\,u \end{array}\right]$$

2.1.1.3 Third Reduction

1. Apply elimination for the last row

$$L_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2(x + tu) & 0 & -2tx & 1 & 1 \end{pmatrix}$$

so that

$$T_7 = L_6 T_6 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & t - \frac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{(8\lambda\,x^2 - 6x)t^2}{3} + \frac{(4\lambda+2)x - 6u^2 - 3)t}{3} - 4x\,u + \frac{2\lambda}{3} + \frac{2}{3} \end{array}\right]$$

2. Scale last row by 3/2

$$S_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

so that

$$T_8 = S_7 T_7 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & t - \frac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 0 & 0 & 0 & 0 & 0 & \text{[alg3]} \end{array}\right]$$

where

$$\text{[alg3]} = \left(4x^2\lambda - 3x\right)t^2 + \left((2\lambda + 1)\,x - 3u^2 - \frac{3}{2}\right)t - 6xu + \lambda + 1$$

The resulting algebraic constraint is [alg3]. Differentiating it gives

$$\frac{\mathrm{d}[\text{alg3}]}{\mathrm{d}t} = (A)x' + (B)u' + (C)\lambda' + (D), \qquad \begin{cases} (A) = (8x\lambda - 3)t^2 + (2\lambda + 1)t - 6u \\ (B) = -6(tu + x) \\ (C) = (4xt + 2)xt + 1 \\ (D) = 8tx^2\lambda + (1 - 6t + 2\lambda)x - 3u^2 - \dfrac{3}{2} \end{cases}$$

reducing the index by one

$$T_9 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & t - \frac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline (A) & 0 & (B) & 0 & (C) & (D) \end{array}\right]$$

2.1.1.4 ODE Form

$(D) \neq 0$ so that DAE is reduced to an ODE.

1. Eliminating in the last row and scaling

$$T_9 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & t - \frac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \hline 0 & 0 & 0 & 0 & 1 & (E)/(C) \end{array}\right]$$

where

$$\begin{aligned}(E) &= (D) - x \cdot (A) - u \cdot (B) \\ &= ((16x\lambda - 9)t + 4\lambda + 3)tu + 16x^2\lambda t + (4\lambda + 3 - 12t)x - \frac{3}{2} - 9u^2\end{aligned}$$

The DAE reduced to ODE become:

$$\begin{cases} x' = u \\ y' = v \\ u' = t - \dfrac{1}{3}(1 + \lambda(1 + 4\,t\,x)) \\ v' = \dfrac{2}{3}(\lambda(1 + t\,x) + 1) - t \\ \lambda' = \dfrac{((16x\lambda - 9)t + 4\lambda + 3)tu + 16x^2\lambda t + (4\lambda + 3 - 12t)x - \frac{3}{2} - 9u^2}{(4xt + 2)xt + 1} \end{cases}$$

The **differential index** is 3 and the invariants are

$$\begin{aligned}[\text{alg1}] &= 1 + y - t\,x^2 = 0 \\ [\text{alg2}] &= x^2 + 2t\,x\,u - v = 0 \\ [\text{alg3}] &= \left(4x^2\lambda - 3x\right)t^2 + \left((2\lambda + 1)\,x - 3u^2 - \frac{3}{2}\right)t - 6xu + \lambda + 1 = 0\end{aligned}$$

notice that

$$\frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} = [\text{alg2}]$$

but

$$\frac{\mathrm{d}[\text{alg2}]}{\mathrm{d}t} \neq [\text{alg3}]$$

and Baumgarte stabilization cannot be used.

2.1.2 Dummy-Variable Reduction

This DAE system exhibits mixed differential and algebraic characteristics. To analyze its structure, we first convert it to semi-explicit form.

Introduce dummy variables:

- $u'(t) = u_{\text{dot}}(t)$

- $v'(t) = v_{\text{dot}}(t)$

The system decomposes into differential (blue) and algebraic (red) components:

$$\begin{cases} x' = u \\ y' = v \\ u' = u_{\text{dot}} \\ v' = v_{\text{dot}} \end{cases} \oplus \begin{cases} 0 = 2u_{\text{dot}} + v_{\text{dot}} + 2tx\lambda - t & \text{(Constraint 1)} \\ 0 = u_{\text{dot}} + 2v_{\text{dot}} - \lambda - 1 + t & \text{(Constraint 2)} \\ 0 = t\,x^2 - y - 1 & \text{(Constraint 3)} \end{cases}$$

where blue equations are the ODE part explicited and the red part are the algebraic constraints. If the algebraic constraints determine a solution for $u_{\text{dot}}(t)$, $v_{\text{dot}}(t)$ and λ the DAE is of index 1 and we can stop here because we can write (formally) the solution of the algebraic part as

- $u_{\text{dot}} = u_{\text{dot}}(x, y, u, v)$

- $v_{\text{dot}} = v_{\text{dot}}(x, y, u, v)$
- $\lambda = \lambda(x, y, u, v)$

2.1.2.1 Variable classification

The variable are classified, in practice the variable that appears with a derivative are differential variable, the other are algebraic:

- **Differential variables**: x, y, u, v
- **Algebraic variables**: λ, u_{dot}, v_{dot}

Notice that algebraic variables appears linearly in the DAE.

2.1.2.2 Reduction Procedure

We attempt to solve for u_{dot} and v_{dot}, but due to the presence of λ, the system is not yet solvable explicitly. We proceed by **differentiating**.

Step 1: Differentiate algebraic constraints

We now differentiate the constraints to reduce the index. Here we want to differentiate **only** the algebraic equation that can reduce the index and do not introduce new differential equation. Here we want to avoid to introduce derivative of the algebraic variable unless is strictly necessary.

We construct a dependency table to identify which constraints to differentiate:

	u_{dot}	v_{dot}	λ
(Constraint 1)	1	1	1
(Constraint 2)	1	1	1
(Constraint 3)	0	0	0

a 0 means the the function do not depend on the variable on the column. A 1 means dependence. (Constraint 3) is the optimal candidate for differentiation as it doesn't introduce new algebraic variables. Differentiate thus (Constraint 3) and substitute known variable in differetial form using the ODE part

$$(\text{Constraint 3})' = x^2 + 2txx' - y' = x^2 + 2txu - v$$

the new equation $(\text{Constraint 3})'$ **is algebraic** and thus is used to replace (Constraint 3). The DAE thus become:

$$\begin{cases} x' = u \\ y' = v \\ u' = u_{\text{dot}} \\ v' = v_{\text{dot}} \end{cases} \oplus \begin{cases} 0 = 2u_{\text{dot}} + v_{\text{dot}} + 2tx\lambda - t & \text{(Constraint 1)} \\ 0 = u_{\text{dot}} + 2v_{\text{dot}} - \lambda - 1 + t & \text{(Constraint 2)} \\ 0 = x^2 + 2txu - v & (\text{Constraint 3})' \\ 0 = t\,x^2 - y - 1 & \text{(Constraint 3)} \end{cases}$$

The red algebraic equations are the new algebraic equation after the derivation. The green algebraic equation is the hidden constraints that is removed from the DAE system but that is satisfied by choosing the right initial conditions.

The red part cannot determine the algebraic variable and must be differentiated again.

Step 2: Differentiate (again) the algebraic constraints

To find the right cadidate(s) here we write a table of the dependencies

	u_{dot}	v_{dot}	λ
(Constraint 1)	1	1	1
(Constraint 2)	1	1	1
$(\text{Constraint 3})'$	0	0	0

$(\text{Constraint 3})'$ is the right candidate because it do not depend on algebraic variable and its derivative do not introduce extra differential equation. Differentiate thus $(\text{Constraint 3})'$ and substitute known variable in differential form using the ODE part

$$\begin{aligned}(\text{Constraint 3})'' &= 2xx' + 2xu + 2tx'u + 2txu' - v' \\ &= 4xu + 2tu^2 + 2txu_{\text{dot}} - v_{\text{dot}}\end{aligned}$$

the new equation $(\text{Constraint 3})''$ **is algebraic** and thus is used to replace $(\text{Constraint 3})'$. The DAE thus become:

$$\begin{cases} x' = u \\ y' = v \\ u' = u_{\text{dot}} \\ v' = v_{\text{dot}} \end{cases} \oplus \begin{cases} 0 = 2u_{\text{dot}} + v_{\text{dot}} + 2tx\lambda - t & \text{(Constraint 1)} \\ 0 = u_{\text{dot}} + 2v_{\text{dot}} - \lambda - 1 + t & \text{(Constraint 2)} \\ 0 = 4xu + 2tu^2 + 2txu_{\text{dot}} - v_{\text{dot}} & (\text{Constraint 3})'' \\ 0 = t\,x^2 - y - 1 & \text{(Constraint 3)} \\ 0 = x^2 + 2txu - v & (\text{Constraint 3})' \end{cases}$$

Te red part is a linear system in the algebraic variable:

$$\begin{pmatrix} 2 & 1 & 2tx \\ 1 & 2 & -1 \\ 2tx & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{\text{dot}} \\ v_{\text{dot}} \\ \lambda \end{pmatrix} = \begin{pmatrix} t \\ 1 - t \\ -2u(2x + tu) \end{pmatrix}$$

Check if solvable

$$\begin{vmatrix} 2 & 1 & 2tx \\ 1 & 2 & -1 \\ 2tx & -1 & 0 \end{vmatrix} = 2tx \begin{vmatrix} 1 & 2tx \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2tx \\ 1 & -1 \end{vmatrix}$$
$$= 2tx(-1 - 4tx) - 2 - 2tx$$
$$= -4tx - 8(tx)^2 - 2$$

Let $z = tx$ then the roots of the polynomial $-2 - 4z - 8z^2$ are

$$z_{1,2} = \frac{-1 - \imath\sqrt{3}}{4}$$

that are complex, thus the linear system is always solvable for tx real. Thus blue part plus red part constitute a DAE of index one. We have done two differentiation and this implies that the original DAE is of **index 3**.

2.1.2.3 ODE Form

The index-1 DAE reduced to ODE become

$$\begin{cases} x' = u \\ y' = v \\ u' = u_{\text{dot}}(t, x, y, u, v) \\ v' = v_{\text{dot}}(t, x, y, u, v) \end{cases}$$

where $u_{\text{dot}}(t, x, y, u, v)$ and $v_{\text{dot}}(t, x, y, u, v)$ are the solution of the linear system

$$\begin{pmatrix} 2 & 1 & 2tx \\ 1 & 2 & -1 \\ 2tx & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{\text{dot}}(t, x, y, u, v) \\ v_{\text{dot}}(t, x, y, u, v) \\ \lambda(t, x, y, u, v) \end{pmatrix} = \begin{pmatrix} t \\ 1 - t \\ -2u(2x + tu) \end{pmatrix} \tag{2.1}$$

there are also two hidden constraints

$$\text{(Constraint 3)} = t\,x^2 - y - 1 = 0,$$
$$\text{(Constraint 3)}' = x^2 + 2txu - v = 0.$$

2.1.2.4 Baumgarte stabilization

For this DAE the hidden constraint can be used to stabilize the algebric constraint: (Constraint 3):

$$\text{(Constraint 3)}'' + 2\eta\omega\text{(Constraint 3)}' + \omega^2\text{(Constraint 3)} = 0$$

that become

$$4xu + 2tu^2 + 2txu_{\text{dot}} - v_{\text{dot}} + 2\eta\omega(x^2 + 2txu - v) + \omega^2(t\,x^2 - y - 1) = 0$$

using this stabilized constraint in the linear system Equation 2.1 become

$$\begin{pmatrix} 2 & 1 & 2tx \\ 1 & 2 & -1 \\ 2tx & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{\text{dot}}(t, x, y, u, v) \\ v_{\text{dot}}(t, x, y, u, v) \\ \lambda(t, x, y, u, v) \end{pmatrix} = \begin{pmatrix} t \\ 1 - t \\ -2u(2x + tu) - (\star) \end{pmatrix} \tag{2.2}$$

$$(\star) = 2\eta\omega(x^2 + 2txu - v) + \omega^2(t\,x^2 - y - 1)$$

2.1.2.5 Jacobians

For implicit solvers we require derivative of algebraic variables. Let

$$A(t,x) = \begin{pmatrix} 2 & 1 & 2tx \\ 1 & 2 & -1 \\ 2tx & -1 & 0 \end{pmatrix},$$

$$X(t,x,y,u,v) = \begin{pmatrix} u_{\text{dot}}(t,x,y,u,v) \\ v_{\text{dot}}(t,x,y,u,v) \\ \lambda(t,x,y,u,v) \end{pmatrix},$$

$$B(t,x,y,u,v) = \begin{pmatrix} t \\ 1-t \\ -2u(2x+tu) - 2\eta\omega(x^2+2txu-v) - \omega^2(t\,x^2-y-1) \end{pmatrix}$$

Deriving $A(t,x)X(t,x,y,u,v) = B(t,x,y,u,v)$

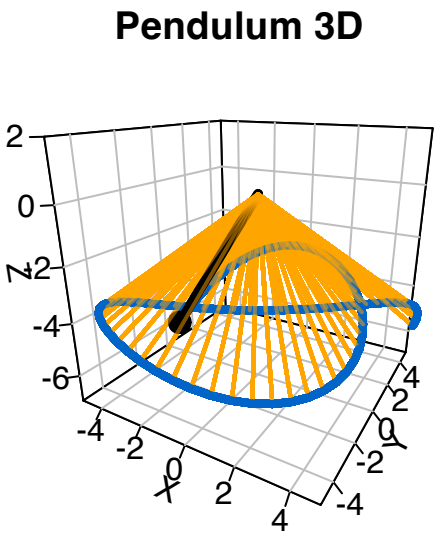
$$A(t,x)\partial_t X(t,x,y,u,v) = \partial_t B(t,x,y,u,v) - \partial_t A(t,x) X(t,x,y,u,v)$$

$$A(t,x)\partial_t X(t,x,y,u,v) = \begin{pmatrix} 1 \\ -1 \\ -2u^2 - 4xu\eta\omega - \omega^2 x^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 2x \\ 0 & 0 & 0 \\ 2x & 0 & 0 \end{pmatrix} X(t,x,y,u,v)$$

$$\partial_t X(t,x,y,u,v) = A(t,x)^{-1}\left(\begin{pmatrix} 1 \\ -1 \\ -2u^2 - 4xu\eta\omega - \omega^2 x^2 \end{pmatrix} - 2x \begin{pmatrix} \lambda(t,x,y,u,v) \\ 0 \\ u_{\text{dot}}(t,x,y,u,v) \end{pmatrix}\right)$$

analogously all the other derivatives can be obtained.

2.2 Example 2: 3D Pendulum



Given the following definitions:

$$T = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) \qquad \text{(Kinetic energy)}$$

$$V = -mgz \qquad \text{(Potential energy)}$$

$$\Phi(x,y,z) = x^2 + y^2 + z^2 - \ell^2 = 0 \qquad \text{(Holonomic constraint)}$$

The constraint enforces that the pendulum mass lies on the surface of a sphere of radius ℓ, centered at the origin.

The corresponding system of Differential-Algebraic Equations (DAE) is:

$$\begin{cases} x'(t) = v_x(t) \\ y'(t) = v_y(t) \\ z'(t) = v_z(t) \\ m\,v_x'(t) = 2\,x(t)\,\lambda(t) \\ m\,v_y'(t) = 2\,y(t)\,\lambda(t) \\ m\,v_z'(t) = 2\,z(t)\,\lambda(t) - mg \\ x^2 + y^2 + z^2 = \ell^2 \end{cases}$$

This is a **DAE of index 3** due to the algebraic constraint $\Phi(x, y, z) = 0$, and the fact that the constraint must be differentiated **three times** to obtain an explicit expression for the highest derivative (λ').

We define

$$Z = \begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \\ \lambda \end{pmatrix}, \quad E(Z,t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G(Z,t) = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \\ \frac{2}{m}x(t)\,\lambda(t) \\ \frac{2}{m}y(t)\,\lambda(t) \\ \frac{2}{m}z(t)\,\lambda(t) - g \\ x^2 + y^2 + z^2 - \ell^2 \end{pmatrix}$$

so that the DAE can be written as

$$E(Z,t)\,Z'(t) = G(Z,t)$$

2.2.1 Index Reduction

The table

$$T = \left[\; E(Z,t) \;\middle|\; G(Z,t) \;\right] = \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & v_x(t) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & v_y(t) \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & v_z(t) \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{m}x(t)\,\lambda(t) \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{2}{m}y(t)\,\lambda(t) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{2}{m}z(t)\,\lambda(t) - g \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 + y^2 + z^2 - \ell^2 \end{array}\right]$$

is used to memorize the operation done on the DAE. Let index $\leftarrow 0$.

1. The algebraic constraint is

$$[\text{alg1}] = x^2 + y^2 + z^2 - \ell^2 = 0$$

2. Reduce by one the index by differentiating [alg1] (index ←= 1)

$$[\text{alg2}] = \frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} = 2\left(xx' + yy' + zz'\right) = 2\left(xv_x + yv_y + zv_z\right)$$

3. Reduce by one the index by differentiating [alg2] (index ←= 2)

$$\begin{aligned}[\text{alg3}] = \frac{\mathrm{d}[\text{alg2}]}{\mathrm{d}t} &= 2\left(x'v_x + y'v_y + z'v_z\right) + 2\left(xv_x' + yv_y' + zv_z'\right) \\ &= 2\left(v_x^2 + v_y^2 + v_z^2\right) + \frac{4\lambda}{m}\left(x^2 + y^2 + z^2\right) - 2gz\end{aligned}$$

Now λ can be solved from [alg3] = 0

$$\lambda = \frac{m}{2}\frac{gz - \left(v_x^2 + v_y^2 + v_z^2\right)}{x^2 + y^2 + z^2}$$

Index-1 DAE

using this solution the index 1 DAE (after the reduction) can be reduced to an ODE by substituting λ as follows:

$$\begin{cases} x' = v_x \\ y' = v_y \\ z' = v_z \\ v_x' = x\left[\dfrac{gz - \left(v_x^2 + v_y^2 + v_z^2\right)}{x^2 + y^2 + z^2}\right] \\ v_y' = y\left[\dfrac{gz - \left(v_x^2 + v_y^2 + v_z^2\right)}{x^2 + y^2 + z^2}\right] \\ v_z' = z\left[\dfrac{gz - \left(v_x^2 + v_y^2 + v_z^2\right)}{x^2 + y^2 + z^2}\right] - mg \end{cases}$$

ODE Form

If you want to include also λ in the reduced ODE derive [alg3] = 0 so that index ←= 3

$$\begin{aligned}\frac{\mathrm{d}[\text{alg3}]}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[2\left(v_x^2 + v_y^2 + v_z^2\right) + \frac{4\lambda}{m}\left(x^2 + y^2 + z^2\right) - 2gz\right] \\ &= 4\left(v_xv_x' + v_yv_y' + v_zv_z'\right) + \frac{4\lambda'}{m}\left(x^2 + y^2 + z^2\right) \\ &+ \frac{8\lambda}{m}\left(xx' + yy' + zz'\right) - 2gz' \\ &= \frac{8\lambda}{m}\left(xv_x + yv_y + zv_z\right) - 4v_zg + \frac{4\lambda'}{m}\left(x^2 + y^2 + z^2\right) \\ &+ \frac{8\lambda}{m}\left(xv_x + yv_y + zv_z\right) - 2v_zg \\ &= \frac{16\lambda}{m}\left(xv_x + yv_y + zv_z\right) - 6v_zg + \frac{4\lambda'}{m}\left(x^2 + y^2 + z^2\right) = 0\end{aligned}$$

Solving for λ'

$$\lambda' = \frac{1}{2}\frac{3\,m\,v_z\,g - 8\lambda\left(xv_x + yv_y + zv_z\right)}{x^2 + y^2 + z^2}$$

The DAE reduced to the ODE become

$$\begin{cases} x' = v_x \\ y' = v_y \\ z' = v_z \\ v_x' = \dfrac{2}{m} x\lambda \\ v_y' = \dfrac{2}{m} y\lambda \\ v_z' = \dfrac{2}{m} z\lambda - g \\ \lambda' = \dfrac{1}{2} \dfrac{3\, m\, v_z\, g - 8\lambda \left(x v_x + y v_y + z v_z \right)}{x^2 + y^2 + z^2} \end{cases}$$

with the hidden constraints

[alg1] $$x^2 + y^2 + z^2 - \ell^2 = 0$$

[alg2] $$2 \left(x v_x + y v_y + z v_z \right) = 0$$

[alg3] $$2 \left(v_x^2 + v_y^2 + v_z^2 \right) + \frac{4\lambda}{m} \left(x^2 + y^2 + z^2 \right) - 2gz = 0$$

2.2.2 Dummy-Variable Reduction

Variable Classification

- **Differential variables**: x, y, z, v_x, v_y, v_z
- **Algebraic variables**: λ

We attempt to solve for λ, the system is not yet solvable explicitly. We proceed by **differentiating**.

Step 1: Differentiate Constraints

Differentiate (Constraint 1) $= x^2 + y^2 + z^2 - \ell^2$:

$$\frac{\mathrm{d}(\text{Constraint 1})}{\mathrm{d}t} = 2xx' + 2yy' + 2zz' = 0 \tag{2.3}$$

Substitute known differential variables:

$$(\text{Constraint 2}) = 2xv_x + 2yv_y + 2zv_z = 0$$

Still no λ. We differentiate once more.

Step 2: Differentiate Again

Differentiate Equation 2.3:

$$\frac{\mathrm{d}(\text{Constraint 2})}{\mathrm{d}t} = 2x'v_x + 2y'v_y + 2z'v_z + 2xv_x' + 2yv_y' + 2zv_z' = 0$$

Substitute known derivatives:

$$(\text{Constraint 3}) = 2(v_x^2 + v_y^2 + v_z^2) + \frac{4\lambda}{m}(x^2 + y^2 + z^2) - 2zg = 0$$

Now λ appears in the DAE. We can solve for λ

$$\lambda = \frac{m}{2} \cdot \frac{zg - (v_x^2 + v_y^2 + v_z^2)}{x^2 + y^2 + z^2}$$

now we can substitute λ in the DAE, but this means reduce the DAE by eliminating a variable. Better define

$$\widetilde{\lambda}(x, y, z, v_x, v_y, v_z) = \frac{m}{2} \cdot \frac{zg - (v_x^2 + v_y^2 + v_z^2)}{x^2 + y^2 + z^2}$$

and obtain the index 1 DAE

$$\begin{cases} x' = v_x \\ y' = v_y \\ z' = v_z \\ v_x' = \dfrac{2}{m} x\,\lambda \\ v_y' = \dfrac{2}{m} y\,\lambda \\ v_z' = \dfrac{2}{m} z\,\lambda - g \\ \lambda = \widetilde{\lambda}(x, y, z, v_x, v_y, v_z) \end{cases}$$

having done two differetiation to obtain an **index 1** DAE he original DAE was of **index 3**.

2.2.3 Constraint Stabilization

At step 4, we obtained the expression for the Lagrange multiplier:

$$\lambda(x, y, z, v_x, v_y, v_z) = \frac{m}{2} \cdot \frac{gz - (v_x^2 + v_y^2 + v_z^2)}{x^2 + y^2 + z^2}.$$

When substituted into the DAE system:

$$\begin{cases} x' = v_x, \\ y' = v_y, \\ z' = v_z, \\ v_x' = \dfrac{2x}{m}\lambda, \\ v_y' = \dfrac{2y}{m}\lambda, \\ v_z' = \dfrac{2z}{m}\lambda - g, \end{cases}$$

this formally reduces the system to an ODE. However, **this does not resolve the numerical drift issue**, as the algebraic constraints are not inherently enforced.

Constraint Dynamics

The Lagrange multiplier λ is derived from the second hidden constraint $[\text{alg3}] = 0$. Defining,

$$\boldsymbol{X} = \begin{pmatrix} x \\ y \\ x \end{pmatrix}, \qquad \boldsymbol{V} = \begin{pmatrix} v_x \\ v_y \\ v_x \end{pmatrix}$$

- position-level constraint

$$[\text{alg1}] = \boldsymbol{X} \cdot \boldsymbol{X} - \ell^2 = 0$$

- velocity-level constraint

$$[\text{alg2}] = 2\boldsymbol{X} \cdot \boldsymbol{V} = 0$$

- acceleration-level constraint

$$[\text{alg3}] = 2\boldsymbol{V} \cdot \boldsymbol{V} + \frac{4\lambda}{m}\boldsymbol{X} \cdot \boldsymbol{X} - 2gz$$

By construction, these constraints satisfy:

$$\frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} = [\text{alg2}], \quad \frac{\mathrm{d}^2[\text{alg1}]}{\mathrm{d}t^2} = \frac{\mathrm{d}[\text{alg2}]}{\mathrm{d}t} = [\text{alg3}].$$

Baumgarte Stabilization

To mitigate drift, we replace $[\text{alg3}] = 0$ with a stabilized ODE:

$$\frac{\mathrm{d}^2[\text{alg1}]}{\mathrm{d}t^2} + 2\eta\omega\frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} + \omega^2[\text{alg1}] = 0,$$

where η (damping ratio) and ω (natural frequency) are tuning parameters. Substituting the constraints:

$$[\text{alg3}] + 2\eta\omega[\text{alg2}] + \omega^2[\text{alg1}] = 0.$$

Expanding this yields:

$$2\boldsymbol{V} \cdot \boldsymbol{V} + \frac{4\lambda}{m}\boldsymbol{X} \cdot \boldsymbol{X} - 2gz + 2\eta\omega \cdot 2\boldsymbol{X} \cdot \boldsymbol{V} + \omega^2(\boldsymbol{X} \cdot \boldsymbol{X} - \ell^2) = 0.$$

Solving for λ

Isolate λ to obtain the stabilized multiplier:

$$\lambda = \frac{m}{4} \cdot \frac{2gz - 2(\boldsymbol{V} \cdot \boldsymbol{V}) - 4\eta\omega(\boldsymbol{X} \cdot \boldsymbol{V}) - \omega^2(\boldsymbol{X} \cdot \boldsymbol{X} - \ell^2)}{\boldsymbol{X} \cdot \boldsymbol{X}}$$

Key Observations

1. **Stabilization Terms**:
   The terms in blue introduce feedback to correct drift:
   - $4\eta\omega(\boldsymbol{X} \cdot \boldsymbol{V})$ penalizes velocity-level deviations.
   - $\omega^2(\boldsymbol{X} \cdot \boldsymbol{X} - \ell^2)$ penalizes position-level deviations.
2. **Parameters**:
   - Choose $\omega \sim 1/\Delta t$ (time step) for critical damping.
   - Set $\eta = 1$ for optimal convergence (overdamping may slow dynamics).
3. **Numerical Behavior**:
   The modified λ ensures constraints are satisfied exponentially, even with numerical errors.

Stabilized ODE

The stabilized system becomes:

$$\begin{cases} x' = v_x, \\ y' = v_y, \\ z' = v_z, \\ v_x' = \dfrac{2x}{m}\lambda_{\text{stab}}(x, y, z, v_x, v_y, v_z), \\ v_y' = \dfrac{2y}{m}\lambda_{\text{stab}}(x, y, z, v_x, v_y, v_z), \\ v_z' = \dfrac{2z}{m}\lambda_{\text{stab}}(x, y, z, v_x, v_y, v_z) - g, \end{cases}$$

where λ_{stab} includes Baumgarte terms:

$$\lambda_{\text{stab}}(x, y, z, v_x, v_y, v_z) = \frac{m}{4} \cdot \frac{2gz - 2(\boldsymbol{V} \cdot \boldsymbol{V}) - 4\eta\omega(\boldsymbol{X} \cdot \boldsymbol{V}) - \omega^2(\boldsymbol{X} \cdot \boldsymbol{X} - \ell^2)}{\boldsymbol{X} \cdot \boldsymbol{X}}$$

This eliminates drift while preserving physical fidelity.

Advantages

- **No Drift**: Constraints are enforced at all levels (position, velocity, acceleration).
- **Tunable Accuracy**: Adjust η, ω to balance stability and numerical stiffness.

Limitations

- Introduces artificial damping if η or ω are too large.
- Requires careful parameter selection for stiff systems.

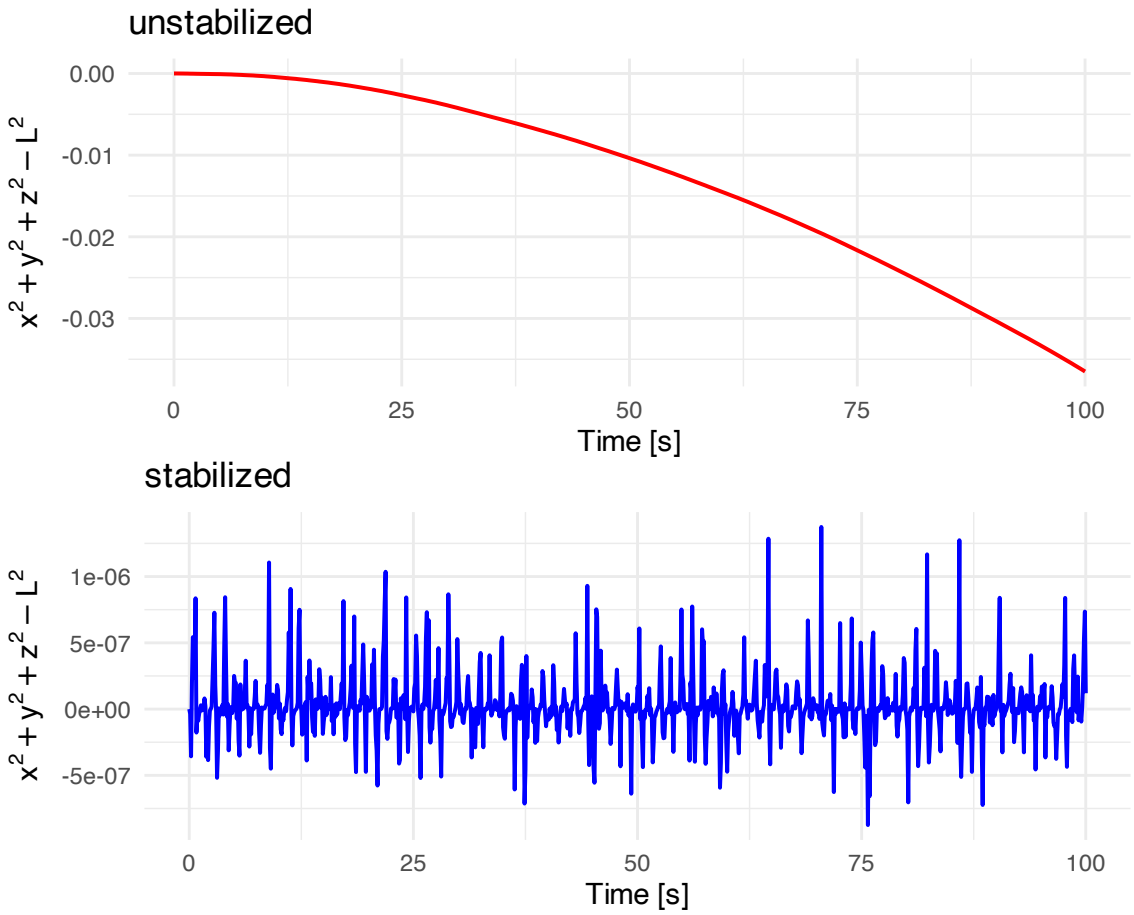
Implementation Note

In practice, use: - $\omega = 10/\Delta t$ (for typical time steps Δt),
- $\eta = 0.7$ (underdamped to avoid overshooting).

This ensures rapid convergence without excessive distortion of the system dynamics.

2.3 Example 4: Double Pendulum

We consider a constrained formulation of a planar double pendulum as a **differential-algebraic equation** (DAE) system:

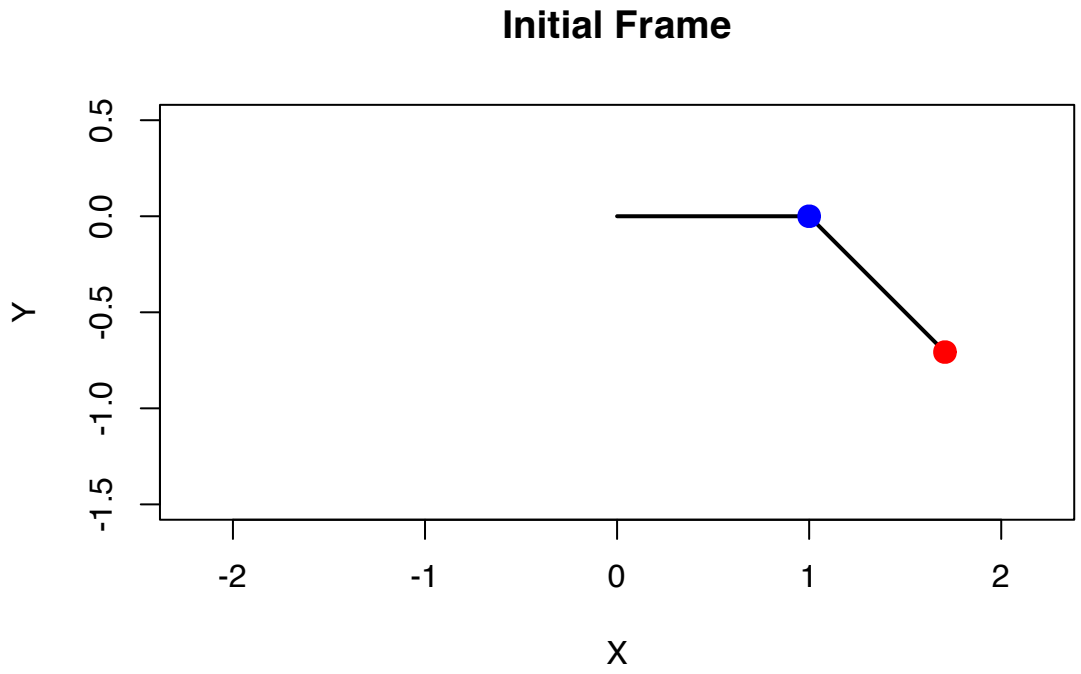


Constraints satisfaction with and without stabilization.

Parameter	Value	Units	Description
Δt	0.1	s	Time step
ω	100	rad/s	Angular frequency
η	0.7	-	Damping coefficient
g	9.81	m/s^2	Gravitational acceleration
m	1	kg	Mass
$x(0) = L$	1	m	Initial position x
$y(0)$	0	m	Initial position y
$z(0)$	0	m	Initial position z
$v_x(0)$	0	m/s	Initial velocity x
$v_y(0)$	0	m/s	Initial velocity y
$v_z(0)$	0	m/s	Initial velocity z

Figure 2.1

$$
\begin{cases}
x_1' = u_1 \\
y_1' = v_1 \\
x_2' = u_2 \\
y_2' = v_2 \\
u_1' = \dfrac{x_1\lambda_1 + (x_1 - x_2)\lambda_2}{m_1} \\
v_1' = \dfrac{y_1\lambda_1 + (y_1 - y_2)\lambda_2}{m_1} - g \\
u_2' = \dfrac{(x_2 - x_1)\lambda_2}{m_2} \\
v_2' = \dfrac{(y_2 - y_1)\lambda_2}{m_2} - g \\
\ell_1^2 = x_1^2 + y_1^2 \\
\ell_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2
\end{cases}
$$



Parameter	Value	Parameter	Value	Parameter	Value
m_1	5	m_2	1	ℓ_1	1
ℓ_2	1	g	9.81	θ_1	$\pi/2$
θ_2	$\pi/4$	ω_1	0	ω_2	-0.1

Figure 2.2: Double pendulum animation with the above parameters

We define the variable vector and matrices:

$$E(Z,t) = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}, \quad Z = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix},$$

$$G(Z,t) = \begin{pmatrix} u_1 \\ v_2 \\ u_2 \\ v_2 \\ \frac{x_1\lambda_1 + (x_1 - x_2)\lambda_2}{m_1} \\ \frac{y_1\lambda_1 + (y_1 - y_2)\lambda_2}{m_1} - g \\ \frac{(x_2 - x_1)\lambda_2}{m_2} \\ \frac{(y_2 - y_1)\lambda_2}{m_2} - g \\ x_1^2 + y_1^2 - \ell_1^2 \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 - \ell_2^2 \end{pmatrix}$$

So that the DAE reads:

$$E(Z,t)\, Z'(t) = G(Z,t)$$

2.3.1 Index Reduction

The table

$$T = \left[\; E(Z,t) \;\middle|\; G(Z,t) \;\right] = \left[\begin{array}{cccccccccc|c} 1 & . & . & . & . & . & . & . & . & . & u_1 \\ . & 1 & . & . & . & . & . & . & . & . & v_2 \\ . & . & 1 & . & . & . & . & . & . & . & u_2 \\ . & . & . & 1 & . & . & . & . & . & . & v_2 \\ . & . & . & . & 1 & . & . & . & . & . & \frac{x_1\lambda_1+(x_1-x_2)\lambda_2}{m_1} \\ . & . & . & . & . & 1 & . & . & . & . & \frac{y_1\lambda_1+(y_1-y_2)\lambda_2}{m_1} - g \\ . & . & . & . & . & . & 1 & . & . & . & \frac{(x_2-x_1)\lambda_2}{m_2} \\ . & . & . & . & . & . & . & 1 & . & . & \frac{(y_2-y_1)\lambda_2}{m_2} - g \\ . & . & . & . & . & . & . & . & . & . & x_1^2 + y_1^2 - \ell_1^2 \\ . & . & . & . & . & . & . & . & . & . & (x_1-x_2)^2 + (y_1-y_2)^2 - \ell_2^2 \end{array}\right]$$

is used to memorize the operation done on the DAE.

R [Semi-Explicit Form] Notice that

$$E(Z,t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

this permits to simplify index reduction working only on the last two rows.

2.3.1.1 First Reduction

1. Derive the algebraic constraint

$$[\text{alg1}] = \begin{pmatrix} x_1^2 + y_1^2 - \ell_1^2 \\ (x_1-x_2)^2 + (y_1-y_2)^2 - \ell_2^2 \end{pmatrix}$$

$$\frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} = \begin{pmatrix} 2x_1x_1' + 2y_1y_1' \\ 2(x_1-x_2)(x_1'-x_2') + 2(y_1-y_2)(y_1'-y_2') \end{pmatrix}$$

2. Update reduced table $\widetilde{T}$ of the last two rows

$$\widetilde{T} \leftarrow \left[\begin{array}{cccccccccc|c} 2x_1 & 2y_1 & . & . & . & . & . & . & . & . & 0 \\ 2(x_1-x_2) & 2(y_1-y_2) & -2(x_1-x_2) & -2(y_1-y_2) & . & . & . & . & . & . & 0 \end{array}\right]$$

apply elimination by subtract a combination of the fist four row to eliminate $2x_1$, $2y_1$ and so on. However this is equivalent to substitute x_1' with u_1, y_1' with v_1 and so on., So that

$$[\text{alg2}] = \frac{\mathrm{d}[\text{alg1}]}{\mathrm{d}t} = \begin{pmatrix} 2x_1u_1 + 2y_1v_1 \\ 2(x_1-x_2)(u_1-u_2) + 2(y_1-y_2)(v_1-v_2) \end{pmatrix}$$

2.3.1.2 Second Reduction

1. Derive the constraint and apply substitution (using $\delta x = x_1 - x_2$, $\delta y = y_1 - y_2$, $\delta u = u_1 - u_2$, $\delta v = v_1 - v_2$)

$$
\begin{aligned}
[\text{alg3}] = \frac{\mathrm{d}[\text{alg2}]}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 2x_1u_1 + 2y_1v_1 \\ 2(x_1 - x_2)(u_1 - u_2) + 2(y_1 - y_2)(v_1 - v_2) \end{pmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 2x_1u_1 + 2y_1v_1 \\ 2\,\delta x\,\delta u_1 + 2\,\delta y\,\delta v \end{pmatrix} \\
&= \begin{pmatrix} 2x_1'u_1 + 2y_1'v_1 + 2x_1u_1' + 2y_1v_1' \\ 2\,\delta x'\,\delta u + 2\,\delta y'\,\delta v + 2\,\delta x\,\delta u' + 2\,\delta y\,\delta v' \end{pmatrix} \\
&= \begin{pmatrix} 2u_1^2 + 2v_1^2 + 2x_1u_1' + 2y_1v_1' \\ 2\,\delta u^2 + 2\,\delta v^2 + 2\,\delta x\,\delta u' + 2\,\delta y\,\delta v' \end{pmatrix}
\end{aligned}
$$

2. defining $1/\widetilde{m} = 1/m_1 + 1/m_2$

$$
\begin{aligned}
\delta u' = u_1' - u_2' &= \frac{x_1\lambda_1 + (x_1 - x_2)\lambda_2}{m_1} - \frac{(x_2 - x_1)\lambda_2}{m_2} \\
&= \frac{x_1\lambda_1}{m_1} + \delta x\lambda_2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \\
&= \frac{x_1\lambda_1}{m_1} + \frac{\delta x\lambda_2}{\widetilde{m}} \\
\delta v' = v_1' - v_2' &= \frac{y_1\lambda_1 + (y_1 - y_2)\lambda_2}{m_1} - g - \frac{(y_2 - y_1)\lambda_2}{m_2} + g \\
&= \frac{y_1\lambda_1}{m_1} + \delta y\lambda_2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \\
&= \frac{y_1\lambda_1}{m_1} + \frac{\delta y\lambda_2}{\widetilde{m}}
\end{aligned}
$$

3. The new constraint become

$$
\begin{aligned}
[\text{alg3}] &= \begin{pmatrix} [\text{alg3}]_1 \\ [\text{alg3}]_2 \end{pmatrix} \\
[\text{alg3}]_1 &= 2(u_1^2 + v_1^2) + \frac{2}{m_1}\lambda_1(x_1^2 + y_1^2) + \frac{2}{m_1}\lambda_2(x_1\delta x + y_1\delta y) - 2gy_1 \\
[\text{alg3}]_2 &= 2(\delta u^2 + \delta v^2) + \frac{2\lambda_1}{m_1}(x_1\delta x + y_1\delta y) + \frac{2\lambda_2}{\widetilde{m}}\left(\delta x^2 + \delta y^2\right)
\end{aligned}
$$

Index-1 DAE

R At this point λ_1 and λ_2 appears. Thus after another derivation the DAE is reduced to an ODE and, thus, the index is 3.

The derivation produce a complex ODE and is not reccomeneded. Instead, λ_1 and λ_2 can be solved, or better written as a function of position and velocity.

Elimination of the multiplier

1. Rewrite [alg3] as a linear system

$$
\frac{1}{m_1}\begin{pmatrix} x_1^2 + y_1^2 & x_1\delta x + y_1\delta y \\ x_1\delta x + y_1\delta y & \frac{m_1}{\widetilde{m}}(\delta x^2 + \delta y^2) \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} gy_1 - (u_1^2 + v_1^2) \\ -(\delta u^2 + \delta v^2) \end{pmatrix}
$$

2. the solution of the linear system are the function

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = m_1 \begin{pmatrix} x_1^2+y_1^2 & x_1\delta x + y_1\delta y \\ x_1\delta x + y_1\delta y & \frac{m_1}{\widetilde{m}}(\delta x^2+\delta y^2)\end{pmatrix}^{-1} \begin{pmatrix} gy_1-(u_1^2+v_1^2) \\ -(\delta u^2+\delta v^2)\end{pmatrix}$$

The final ODE is

$$\begin{cases} x_1' = u_1 \\ y_1' = v_1 \\ x_2' = u_2 \\ y_2' = v_2 \\ u_1' = \dfrac{x_1}{m_1}\lambda_1(x_1,y_1,x_2,y_2,u_1,v_1,u_2,v_2) + \dfrac{(x_1-x_2)}{m_1}\lambda_2(x_1,y_1,x_2,y_2,u_1,v_1,u_2,v_2) \\ v_1' = \dfrac{y_1}{m_1}\lambda_1(x_1,y_1,x_2,y_2,u_1,v_1,u_2,v_2) + \dfrac{(y_1-y_2)}{m_1}\lambda_2(x_1,y_1,x_2,y_2,u_1,v_1,u_2,v_2) - g \\ u_2' = \dfrac{(x_2-x_1)}{m_2}\lambda_2(x_1,y_1,x_2,y_2,u_1,v_1,u_2,v_2) \\ v_2' = \dfrac{(y_2-y_1)}{m_2}\lambda_2(x_1,y_1,x_2,y_2,u_1,v_1,u_2,v_2) - g \end{cases}$$

Drift of the constraints. Notice that scale is logarithmic and error grows exponentially.

Baumgarte Stabilization

To stabilize the index-3 DAE system of the double pendulum and avoid constraint drift during numerical integration, we introduce Baugarte stabilization, a method that replaces the original constraints with second-order damped versions of them.

We define the stabilized constraints by enforcing a second-order differential equation for each component:

$$\frac{\mathrm{d}^2[\mathrm{alg1}]}{\mathrm{d}t^2} + 2\eta\omega\frac{\mathrm{d}[\mathrm{alg1}]}{\mathrm{d}t} + \omega^2[\mathrm{alg1}] = \mathbf{0}$$

or equivalently

$$[\mathrm{alg3}] + 2\eta\omega[\mathrm{alg2}] + \omega^2[\mathrm{alg1}] = \mathbf{0}$$

that become

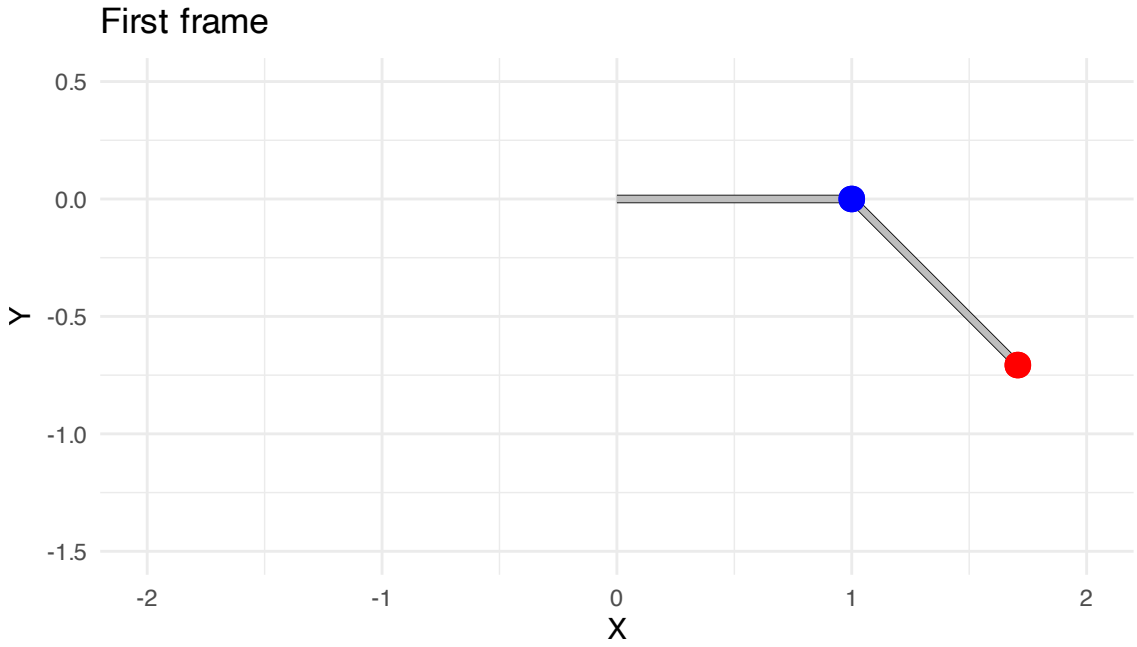
$$2(u_1^2+v_1^2) + \frac{2}{m_1}\lambda_1(x_1^2+y_1^2) + \frac{2}{m_1}\lambda_2(x_1\delta x + y_1\delta y) - 2gy_1 + 2\eta\omega(2x_1u_1+2y_1v_1) + \omega^2(x_1^2+y_1^2-\ell_1^2) = 0$$

$$2(\delta u^2+\delta v^2) + \frac{2\lambda_1}{m_1}(x_1\delta x + y_1\delta y) + \frac{2\lambda_2}{\widetilde{m}}\left(\delta x^2+\delta y^2\right) + 2\eta\omega(2\delta x\,\delta u + 2\delta y\,\delta v) + \omega^2(\delta x^2+\delta y^2-\ell_2^2) = 0$$

that become the linear system

$$\frac{1}{m_1}\begin{pmatrix} x_1^2+y_1^2 & x_1\delta x + y_1\delta y \\ x_1\delta x + y_1\delta y & \frac{m_1}{\widetilde{m}}(\delta x^2+\delta y^2)\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix} r_1 \\ r_2\end{pmatrix}$$

where



Parameter	Parameter	Parameter
$m_1 = 5$	$m_2 = 1$	$\ell_1 = 1$
$\ell_2 = 1$	$g = 9.81$	$\theta_1 = \pi/2$
$\theta_2 = \pi/4$	$\omega_1 = 0$	$\omega_2 = -0.1$
$x_1(0) = \ell_1 \sin(\theta_1)$	$y_1(0) = -\ell_1 \cos(\theta_1)$	$x_2(0) = x_1(0) + \ell_2 \sin(\theta_2)$
$y_2(0) = x_2(0) - \ell_2 \cos(\theta_2)$	$u_1(0) = 0$	$v_1(0) = 0$
$u_2(0) = 0$	$v_2(0) = 0$	

Figure 2.3: Parameters for the double pendulum solution of the reduced DAE without stabilization

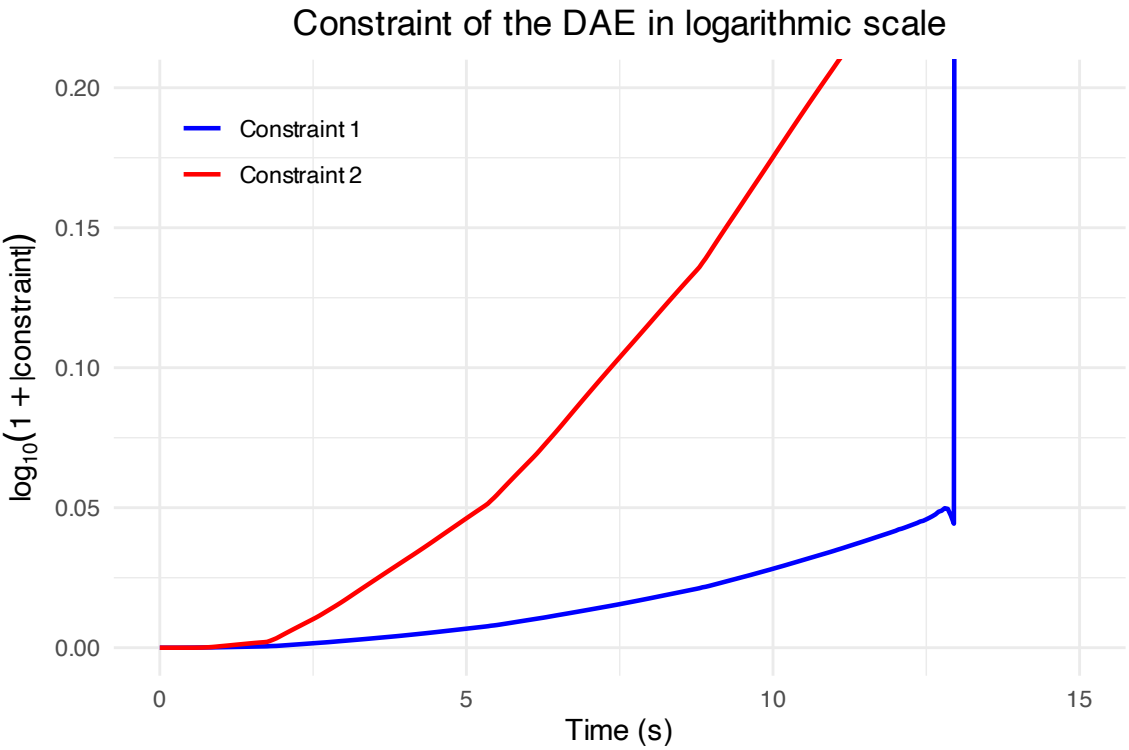


Figure 2.4: Constraint drift in the unstabilized double-pendulum simulation.

$$r_1 = g y_1 - (u_1^2 + v_1^2) - 2\eta\omega(x_1 u_1 + y_1 v_1) - \frac{1}{2}\omega^2(x_1^2 + y_1^2 - \ell_1^2)$$

$$r_2 = -(\delta u^2 + \delta v^2) - 2\eta\omega(\delta x\, \delta u + \delta y\, \delta v) - \frac{1}{2}\omega^2(\delta x^2 + \delta y^2 - \ell_2^2)$$

Parameters for the double pendulum solution of the reduced DAE with stabilization

Parameter	Parameter
$m_1 = 5$	$m_2 = 1$
$\ell_1 = 1$	$\ell_2 = 1$
$g = 9.81$	$\theta_1 = \pi/2$
$\theta_2 = \pi/4$	$\omega_1 = 0$
$\omega_2 = -0.1$	$x_1(0) = \ell_1 \sin(\theta_1)$
$y_1(0) = -\ell_1 \cos(\theta_1)$	$x_2(0) = x_1(0) + \ell_2 \sin(\theta_2)$
$y_2(0) = x_2(0) - \ell_2 \cos(\theta_2)$	$u_1(0) = 0$
$v_1(0) = 0$	$u_2(0) = 0$
$v_2(0) = 0$	$\eta = 0.7$
$\Delta t = 0.05$	$\omega = 2.4/\Delta t$

Drift of the constraints. Notice that drift do not grow and is maintained close to zero by Baumgarte stabilization.

2.3.2 Dummy-Variable Reduction

Variable Classification

- **Differential variables**: x_1, y_1, x_2, y_2, u_1, v_1, u_2, v_2
- **Algebraic variables**: λ_1, λ_2.

We attempt to solve for λ_1 and λ_2, but the system is not yet solvable explicitly. We proceed by **differentiating**.

Step 1: Differentiate Constraints

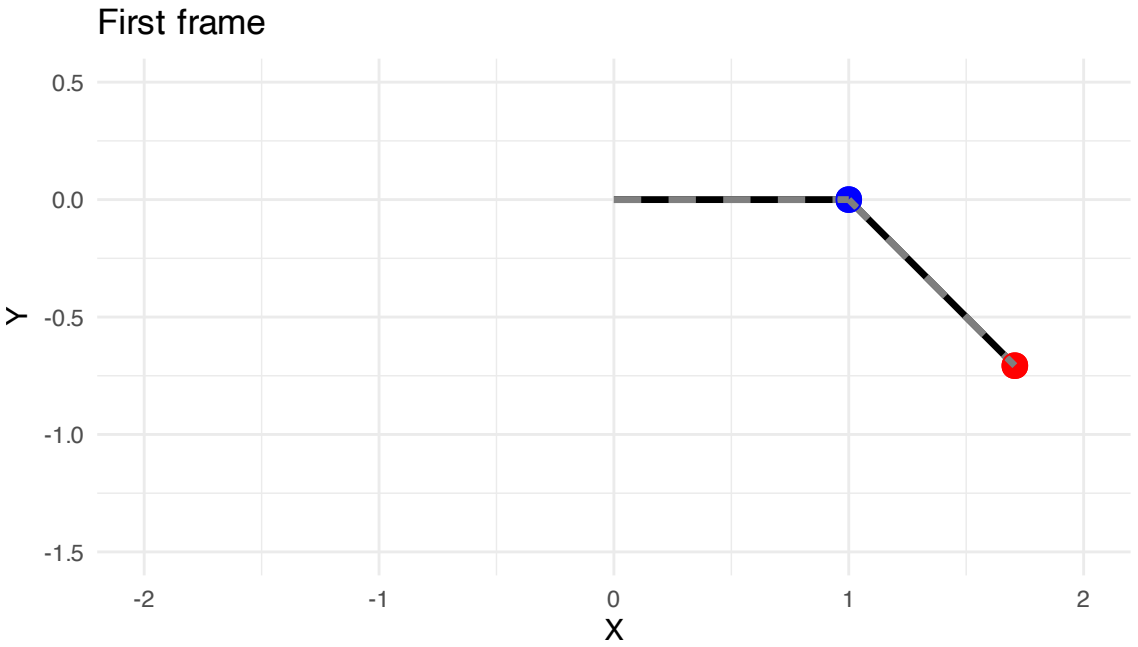


Figure 2.5: Double-pendulum solution obtained with Baumgarte stabilization.

$$
\begin{aligned}
(\text{Constraint } 1)' &= 2x_1x_1' + 2y_1y_1' = 2(x_1u_1 + y_1v_1) \\
(\text{Constraint } 2)' &= 2(x_1 - x_2)(x_1' - x_2') + 2(y_1 - y_2)(y_1' - y_2') \\
&= 2((x_1 - x_2)(u_1 - u_2) + (y_1 - y_2)(v_1 - v_2))
\end{aligned}
$$

Substitute known differential variables and use shortcuts $\delta x = x_1 - x_2$, $\delta y = y_1 - y_2$, $\delta u = u_1 - u_2$, $\delta v = v_1 - v_2$ so that the DAE become:

$$
\begin{cases}
x_1' = u_1 \\
y_1' = v_1 \\
x_2' = u_2 \\
y_2' = v_2 \\
u_1' = \dfrac{x_1\lambda_1 + (x_1 - x_2)\lambda_2}{m_1} \\
v_1' = \dfrac{y_1\lambda_1 + (y_1 - y_2)\lambda_2}{m_1} - g \\
u_2' = \dfrac{(x_2 - x_1)\lambda_2}{m_2} \\
v_2' = \dfrac{(y_2 - y_1)\lambda_2}{m_2} - g
\end{cases}
\oplus
\begin{cases}
0 = 2(x_1u_1 + y_1v_1) & (\text{Constraint } 1)' \\
0 = 2(\delta x\delta u + \delta y\delta v) & (\text{Constraint } 2)' \\
\ell_1^2 = x_1^2 + y_1^2 & (\text{Constraint } 1) \\
\ell_2^2 = \delta x^2 + \delta y^2 & (\text{Constraint } 2)
\end{cases}
$$

The red algebraic equations are the new algebraic equation after the derivation. The green algebraic equation is the hidden constraints that is removed from the DAE system but that is satisfied by choosing the right initial conditions.

The red part cannot determine the algebraic variable and must be differentiated again.

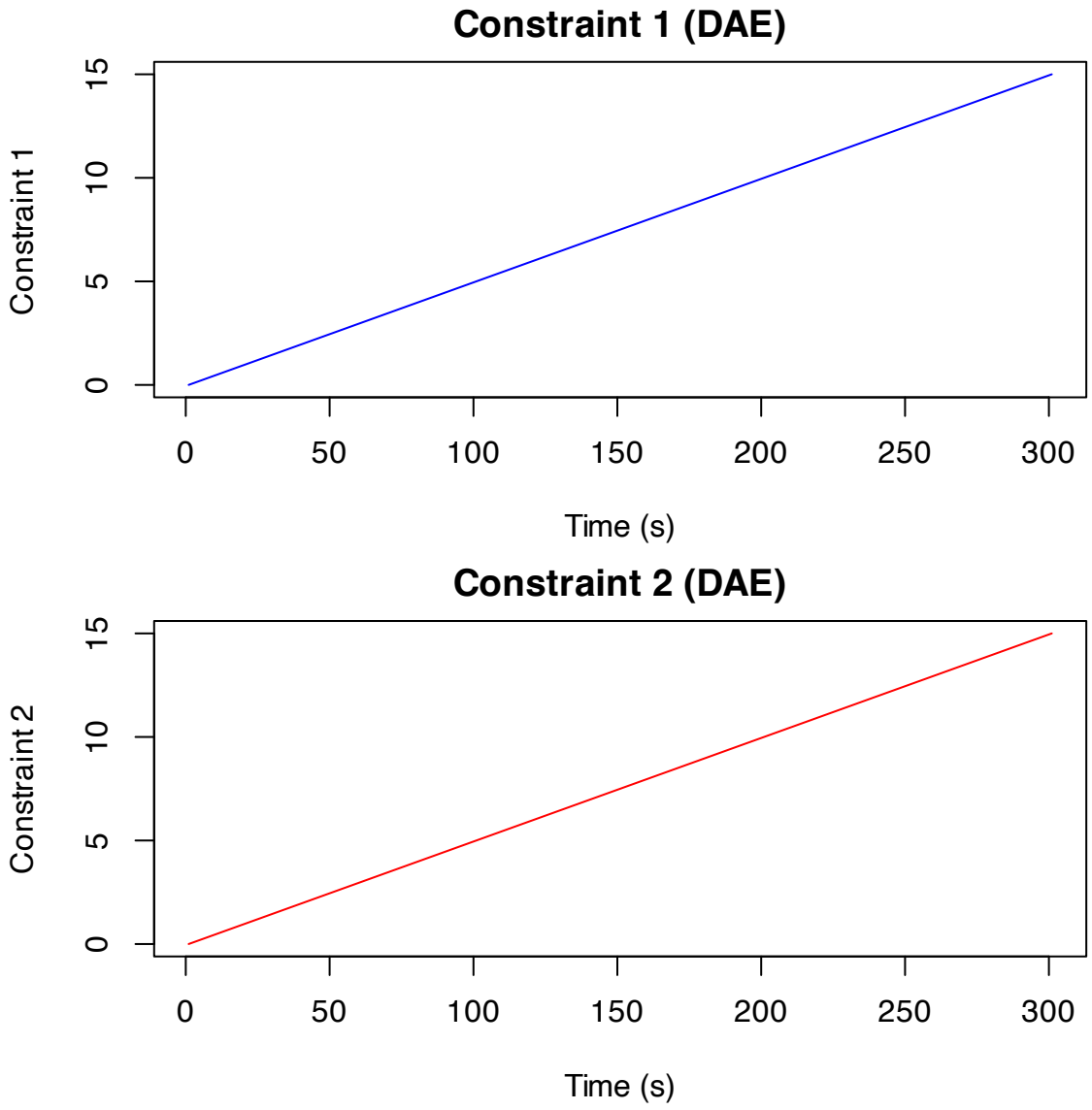


Figure 2.6: Constraint residuals controlled by Baumgarte stabilization.

Step 2: Differentiate Again

To find the right candidate(s) here we write a table of the dependencies

	λ_1	λ_2
$(\text{Constraint } 1)'$	0	0
$(\text{Constraint } 2)'$	0	0

Both constraints are candidate for differentiation as it doesn't introduce new algebraic variables. Differentiate thus the constraints and substitute known variable in differential form using the ODE part

Step 3: Differentiate Constraints

$$(\text{Constraint } 1)'' = 2x_1'u_1 + 2y_1'v_1 + 2x_1 + 2y_1v_1' = 0$$
$$(\text{Constraint } 2)'' = 2\delta x'\delta u + 2\delta y'\delta v + 2\delta x\delta u' + 2\delta y\delta v' = 0$$

Substitute known differential variables

$$2(u_1^2 + v_1^2) + 2\frac{x_1^2\lambda_1 + x_1\delta x\lambda_2}{m_1} + 2\frac{y_1^2\lambda_1 + y_1\delta y\lambda_2}{m_1} - 2y_1g = 0$$
$$2(\delta u^2 + \delta v^2) + 2\delta x\delta u' + 2\delta y\delta v' = 0$$

using $1/\widetilde{m} = 1/m_1 + 1/m_2$

$$\delta u' = \frac{x_1\lambda_1 + \delta x\lambda_2}{m_1} + \frac{\delta x\lambda_2}{m_2} = \frac{x_1}{m_1}\lambda_1 + \frac{\delta x}{\widetilde{m}}\lambda_2$$
$$\delta v' = \frac{y_1\lambda_1 + \delta y\lambda_2}{m_1} + \frac{\delta y\lambda_2}{m_2} = \frac{y_1}{m_1}\lambda_1 + \frac{\delta y}{\widetilde{m}}\lambda_2$$

so that constraints after substitution become

$$(\text{Constraint } 1)'' = 2(u_1^2 + v_1^2) + \frac{2}{m_1}((x_1^2 + y_1^2)\lambda_1 + (x_1\delta x + y_1\delta y)\lambda_2) - 2y_1g = 0$$
$$(\text{Constraint } 2)'' = 2(\delta u^2 + \delta v^2) + 2\delta x\left(\frac{x_1}{m_1}\lambda_1 + \frac{\delta x}{\widetilde{m}}\lambda_2\right) + 2\delta y\left(\frac{y_1}{m_1}\lambda_1 + \frac{\delta y}{\widetilde{m}}\lambda_2\right) = 0$$

and now λ_1 and λ_2 appears linearly:

$$\begin{pmatrix} x_1^2 + y_1^2 & x_1\delta x + y_1\delta y \\ x_1\delta x + y_1\delta y & \frac{m_1}{\widetilde{m}}(\delta x^2 + \delta y^2) \end{pmatrix}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = m_1\begin{pmatrix} y_1g - (u_1^2 + v_1^2) \\ -(\delta u^2 + \delta v^2)\end{pmatrix}$$

2.3.2.1 Index-1 DAE

Thus DAE now is of index 1 and can be written as:

$$\begin{cases} x_1' = u_1 \\ y_1' = v_1 \\ x_2' = u_2 \\ y_2' = v_2 \\ u_1' = \dfrac{x_1\lambda_1 + (x_1 - x_2)\lambda_2}{m_1} \\ v_1' = \dfrac{y_1\lambda_1 + (y_1 - y_2)\lambda_2}{m_1} - g \\ u_2' = \dfrac{(x_2 - x_1)\lambda_2}{m_2} \\ v_2' = \dfrac{(y_2 - y_1)\lambda_2}{m_2} - g \\ \begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = \boldsymbol{A}(t, Z)^{-1}\boldsymbol{b}(t, Z) \end{cases}$$

where

$$Z = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}, \qquad \boldsymbol{A}(t, Z) = \begin{pmatrix} x_1^2 + y_1^2 & x_1\delta x + y_1\delta y \\ x_1\delta x + y_1\delta y & \frac{m_1}{\widetilde{m}}(\delta x^2 + \delta y^2) \end{pmatrix},$$

$$\boldsymbol{b}(t, Z) = m_1 \begin{pmatrix} y_1 g - (u_1^2 + v_1^2) \\ -(\delta u^2 + \delta v^2) \end{pmatrix}$$

Moreover using the right initial condition satisfy the hidden constraints.

$$\begin{cases} 0 = x_1^2 + y_1^2 - \ell_1^2 & \text{(Constraint 1)} \\ 0 = \delta x^2 + \delta y^2 - \ell_2^2 & \text{(Constraint 2)} \\ 0 = 2(x_1u_1 + y_1v_1) & \text{(Constraint 1)}' \\ 0 = 2(\delta x\delta u + \delta y\delta v) & \text{(Constraint 2)}' \end{cases}$$

2.3.2.2 Baumgarte Stabilization

For this DAE the hidden constraints can be used to stabilize the algebric constraints: (Constraint 1) and (Constraint 2)

$$\text{(Constraint 1)}_{\text{stab}} = \text{(Constraint 1)}'' + 2\eta_1\omega_1\text{(Constraint 1)}' + \omega_1^2\text{(Constraint 1)} = 0$$
$$\text{(Constraint 2)}_{\text{stab}} = \text{(Constraint 2)}'' + 2\eta_2\omega_2\text{(Constraint 2)}' + \omega_2^2\text{(Constraint 2)} = 0$$

and substituting (diving also by 2)

$$(\text{Constraint 1})_{\text{stab}} = u_1^2 + v_1^2 + \frac{(x_1^2 + y_1^2)\lambda_1 + (x_1\delta x + y_1\delta y)\lambda_2}{m_1} - y_1 g + (\star)_1$$

$$(\text{Constraint 2})_{\text{stab}} = \delta u^2 + \delta v^2 + \frac{(x_1\delta x + y_1\delta y)\lambda_1}{m_1} + \frac{(\delta x^2 + \delta y^2)\lambda_2}{\widetilde{m}} + (\star)_2$$

$$(\star)_1 = 2\eta_1\omega_1(x_1u_1 + y_1v_1) + \omega_1^2\frac{x_1^2 + y_1^2 - \ell_1^2}{2}$$

$$(\star)_2 = 2\eta_2\omega_2(\delta x\delta u + \delta y\delta v) + \omega_2^2\frac{\delta x^2 + \delta y^2 - \ell_2^2}{2}$$

and using the stabilized constraints

$$\boldsymbol{b}_{\text{stab}}(t, Z) = m_1\begin{pmatrix} y_1 g - (u_1^2 + v_1^2) - (\star)_1 \\ -(\delta u^2 + \delta v^2) - (\star)_2 \end{pmatrix}$$

2.4 Example 7: Circuit DAE

From [3]

2.4.1 Circuit Model

We start with a system of differential-algebraic equations (DAE) that describes a circuit. The equations are given by:

$$\begin{pmatrix} C & -C & 0 \\ -C & C & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1' \\ v_3' \\ i \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1/R & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_3 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V(t) \end{pmatrix}$$

2.4.1.1 Semi-Explicit Form

We decompose the system into differential (blue) and algebraic (red) components as follows:

$$\begin{cases} v_1' = v_{1,\text{dot}} \\ v_3' = v_{3,\text{dot}} \end{cases} \oplus \begin{cases} C(v_{1,\text{dot}} - v_{3,\text{dot}}) - i = 0, & \text{(Constraint 1)} \\ RC(v_{1,\text{dot}} - v_{3,\text{dot}}) - v_3 = 0, & \text{(Constraint 2)} \\ v_1 + V(t) = 0, & \text{(Constraint 3)} \end{cases}$$

In this system:

- **Differential variables**: v_1, v_3.
- **Algebraic variables**: $v_{1,\text{dot}}$, $v_{3,\text{dot}}$, i.

We begin by attempting to solve for the variables ($v_{1,\text{dot}}$, $v_{3,\text{dot}}$, and i), but the system is not explicitly solvable at this stage. We need to reduce the index of the system. To do this, we differentiate the algebraic constraints.

2.4.1.2 Reduction Procedure

We attempt to solve for $v_{1,\text{dot}}$, $v_{3,\text{dot}}$ and i_{dot} but the system is not yet solvable explicitly. We proceed by **differentiating**.

Step 1: Differentiate algebraic constraints

We now differentiate the constraints to reduce the index. Here we want to differentiate **only** the algebraic equation that can reduce the index and do not introduce new differential equation. Here we want to avoid to introduce derivative of the algebraic variable unless is strictly necessary.

We construct a dependency table to identify which constraints to differentiate:

	$v_{1,\text{dot}}$	$v_{3,\text{dot}}$	i
(Constraint 1)	1	1	1
(Constraint 2)	1	1	0
(Constraint 3)	0	0	0

A 0 means no dependence on the corresponding variable, while a 1 indicates dependence. Notice that Constraint 3 does not depend on $v_{1,\text{dot}}$, $v_{3,\text{dot}}$, or i, making it the most suitable candidate for differentiation.

We differentiate Constraint 3:

$$(\text{Constraint 3})' = v_1' - V'(t) = v_{1,\text{dot}} - V'(t)$$

After differentiation, the system becomes:

$$\begin{cases} v_1' = v_{1,\text{dot}} \\ v_3' = v_{3,\text{dot}} \end{cases} \oplus \begin{cases} C(v_{1,\text{dot}} - v_{3,\text{dot}}) - i = 0, & \text{(Constraint 1)} \\ RC(v_{1,\text{dot}} - v_{3,\text{dot}}) - v_3 = 0, & \text{(Constraint 2)} \\ v_{1,\text{dot}} + V'(t) = 0, & (\text{Constraint 3})' \\ v_1 + V(t) = 0, & \text{(Constraint 3)} \end{cases}$$

Here, the red equations represent the newly derived algebraic equations, while the green equation represents the original constraint that has now been removed from the system. This hidden constraint must still be satisfied, but it can be accounted for through proper initial conditions.

Step 2: Differentiate (again) the algebraic constraints

To find the right candidate(s) here we write a table of the dependencies

	$v_{1,\text{dot}}$	$v_{3,\text{dot}}$	i
(Constraint 1)	1	1	1
(Constraint 2)	1	1	0
$(\text{Constraint 3})'$	1	0	0

None of the constraints can be differentiated further at this stage. Therefore, we check the Jacobian of the system to determine whether it is singular. If the Jacobian is non-singular, we can solve the system and the DAE will be reduced to index-1.

We analyze the sparsity pattern of the Jacobian matrix:

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}$$

This structure suggests that the matrix is non-singular, but we will use MATLAB's sprank function to check for its rank:

```
sparsity_pattern = [
  1 1 1;  % Constraint 1
  1 1 0;  % Constraint 2
  1 0 0]; % Constraint 3'

fprintf('Estimated rank = %d\n', sprank(sparsity_pattern) )

> Estimated rank = 3
```

Since the matrix is structurally non-singular, we proceed by solving for the derivatives:

$$\begin{aligned} v_{1,\text{dot}} &= -V'(t) \\ v_{3,\text{dot}} &= v_{1,\text{dot}} - v_3/(RC) = -V'(t) - v_3/(RC) \\ i &= C(v_{1,\text{dot}} - v_{3,\text{dot}}) = v_3/R \end{aligned}$$

and thus the DAE is of index-1. The DAE is better written as

$$\begin{cases} v_1' = v_{1,\text{dot}} \\ v_3' = v_{3,\text{dot}} \end{cases} \oplus \begin{cases} \begin{pmatrix} C & -C & -1 \\ RC & -RC & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{1,\text{dot}} \\ v_{2,\text{dot}} \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ -V'(t) \end{pmatrix}, \\ \\ v_1 + V(t) = 0, \quad \text{(Constraint 3)} \end{cases}$$

where the green part are the hidden constraints that are satisfied by choosing the correct initial conditions.

Note that we have done one derivation on the constraints and the DAE is of index 1, thus, the original DAE is of index 2.

2.4.2 Baumgarte Stabilization

To avoid drift in the hidden constraints, we apply **Baumgarte stabilization**:

$$(\text{Constraints 3})' + \frac{1}{\tau}(\text{Constraints 3}) = 0$$

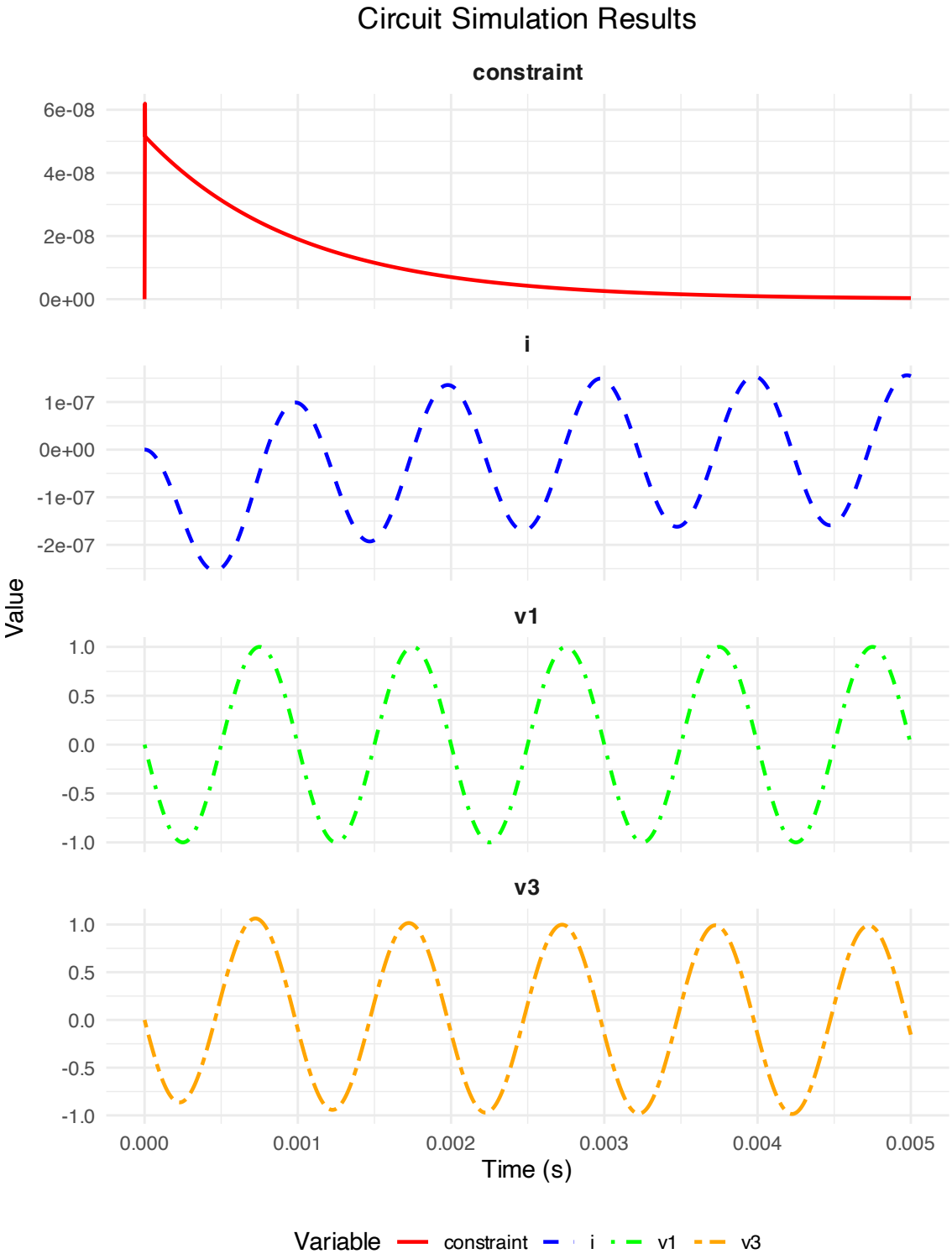
and substituting the expression

$$v_{1,\text{dot}} + V'(t) + \frac{v_1 + V(t)}{\tau} 0$$

The stabilized constraints are:

$$\begin{pmatrix} C & -C & -1 \\ RC & -RC & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{1,\text{dot}} \\ v_{2,\text{dot}} \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ -V'(t) - (v_1 + V(t))/\tau \end{pmatrix}$$

2.4.3 Simulation Results



2.5 Example 8: Index-5 Crane

Example taken from [2] example 2.1.

2.5.1 Crane DAE

We consider an index-five DAE system that models the planar motion of a crane.
The goal is to determine the horizontal velocity u_1 and the angular velocity u_2 of a winch of

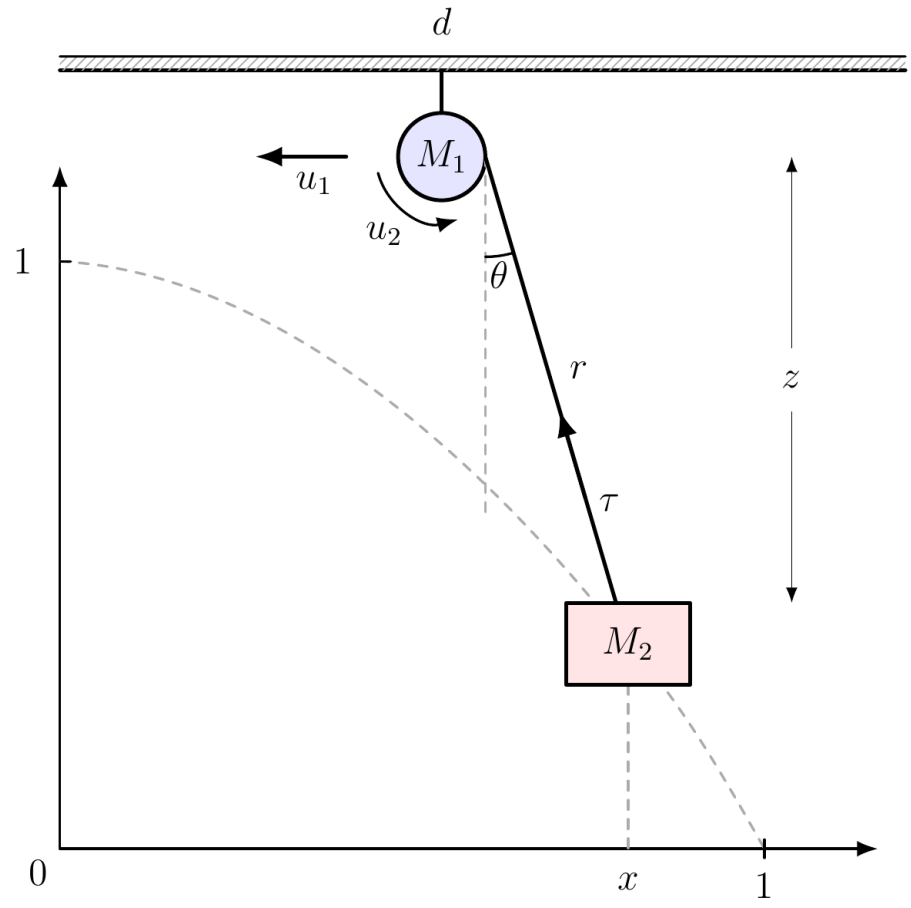


Figure 2.7: Physical model of the DAE

mass M_1, so that the attached load of mass M_2 follows a prescribed path $p_1(t)$ and $p_2(t)$.

The DAEs, derived from Newton's second law, are given by:

$$\begin{cases} x' = v_x \\ z' = v_z \\ d' = v_d \\ r' = v_r \\ M_2 v_x' = -\tau \sin\theta \\ M_2 v_z' = -\tau \cos\theta + mg \\ M_1 v_d' = -C_1 v_d + u_1 + \tau \sin\theta \\ J v_r' = -C_2 v_r - C_3 u_2 + C_3^2 \tau \\ x = r \sin\theta + d \\ z = r \cos\theta \\ x = p_1(t) \\ z = p_2(t) \end{cases}$$

where:

- (x, z) denotes the position of the load,
- τ is the tension in the cable,
- θ is the angle of the cable with the vertical,
- J is the moment of inertia of the winch,
- d is the position of the trolley,
- r is the length of the cable.

The constants satisfy $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, $g > 0$, and $m > 0$.

2.5.1.1 Semi-Explicit Form

For the previous DAE system introduce dummy variables

- $v_{x,\text{dot}}$
- $v_{z,\text{dot}}$
- $v_{d,\text{dot}}$
- $v_{r,\text{dot}}$

and the system decomposes into differential (blue) and algebraic (red) components:

$$\begin{cases} x' = v_x \\ z' = v_z \\ d' = v_d \\ r' = v_r \\ v_x' = v_{x,\text{dot}} \\ v_z' = v_{z,\text{dot}} \\ v_d' = v_{d,\text{dot}} \\ v_r' = v_{r,\text{dot}} \end{cases} \oplus \begin{cases} 0 = M_2 v_{x,\text{dot}} + \tau \sin\theta & \text{(Constraint 1)} \\ 0 = M_2 v_{z,\text{dot}} + \tau \cos\theta - mg & \text{(Constraint 2)} \\ 0 = M_1 v_{d,\text{dot}} + C_1 v_d - u_1 - \tau \sin\theta & \text{(Constraint 3)} \\ 0 = J v_{r,\text{dot}} + C_2 v_r + C_3 u_2 - C_3^2 \tau & \text{(Constraint 4)} \\ 0 = r \sin\theta + d - x & \text{(Constraint 5)} \\ 0 = r \cos\theta - z & \text{(Constraint 6)} \\ 0 = p_1(t) - x & \text{(Constraint 7)} \\ 0 = p_2(t) - z & \text{(Constraint 8)} \end{cases}$$

For this DAE we have:

- **Differential variables**: x, z, d, r, v_x, v_z, v_d, v_r .
- **Algebraic variables**:- $v_{x,\text{dot}}$, $v_{z,\text{dot}}$, $v_{d,\text{dot}}$, $v_{r,\text{dot}}$, u_1, u_2. τ, θ

The algebraic variables do not appears linearly in the constraints. In particular θ appears nonlinear as $\sin\theta$ and $\cos\theta$. Introducing the new dummy variable $\theta_{\text{dot}} = \theta'$ we have for the differential part:

$$\begin{cases} x' = v_x \\ z' = v_z \\ d' = v_d \\ r' = v_r \\ v_x' = v_{x,\text{dot}} \\ v_z' = v_{z,\text{dot}} \\ v_d' = v_{d,\text{dot}} \\ v_r' = v_{r,\text{dot}} \\ \theta' = \theta_{\text{dot}} \end{cases}$$

and now

- **Differential variables**: x, z, d, r, v_x, v_z, v_d, v_r, θ.
- **Algebraic variables**:- $v_{x,\text{dot}}$, $v_{z,\text{dot}}$, $v_{d,\text{dot}}$, $v_{r,\text{dot}}$, u_1, u_2. τ, θ_{dot}

and not constraints are linear in the algebraic variables.

2.5.1.2 Reduction Procedure

We attempt to solve for $v_{x,\text{dot}}$, $v_{z,\text{dot}}$, $v_{d,\text{dot}}$, $v_{r,\text{dot}}$, but the system is not yet solvable explicitly. We proceed by **differentiating**.

Step 1: Differentiate algebraic constraints

We now differentiate the constraints to reduce the index. Here we want to differentiate **only** the algebraic equation that can reduce the index and do not introduce new differential equation. Here we want to avoid to introduce derivatove of the algebraic variable unless is strictly necessary.

We construct a dependency table to identify which constraints to differentiate:

	$v_{x,\text{dot}}$	$v_{z,\text{dot}}$	$v_{d,\text{dot}}$	$v_{r,\text{dot}}$	u_1	u_2	τ	θ_{dot}
(Constraint 1)	1	0	0	0	0	0	1	0
(Constraint 2)	0	1	0	0	0	0	1	0
(Constraint 3)	0	0	1	0	1	0	1	0
(Constraint 4)	0	0	0	1	0	1	1	0
(Constraint 5)	0	0	0	0	0	0	0	0
(Constraint 6)	0	0	0	0	0	0	0	0
(Constraint 7)	0	0	0	0	0	0	0	0
(Constraint 8)	0	0	0	0	0	0	0	0

a 0 means the the function do not depend on the variable on the column. A 1 means dependence. The last four constraints are candidate for differentiation as it doesn't introduce

new algebraic variables. Differentiate thus the last constraints and substitute known variable in differential form using the ODE part

$$\begin{aligned}
&\text{(Constraint 5)}' \quad 0 = v_r \sin\theta + r(\cos\theta)\theta_{\text{dot}} + v_d - v_x \\
&\text{(Constraint 6)}' \quad 0 = v_r \cos\theta - r(\sin\theta)\theta_{\text{dot}} - v_z \\
&\text{(Constraint 7)}' \quad 0 = p_1'(t) - v_x \\
&\text{(Constraint 8)}' \quad 0 = p_2'(t) - v_z
\end{aligned}$$

so that algebraic parte of the DAE become:

$$\begin{cases}
0 = M_2 v_{x,\text{dot}} + \tau \sin\theta & \text{(Constraint 1)} \\
0 = M_2 v_{z,\text{dot}} + \tau \cos\theta - mg & \text{(Constraint 2)} \\
0 = M_1 v_{d,\text{dot}} + C_1 v_d - u_1 - \tau \sin\theta & \text{(Constraint 3)} \\
0 = J v_{r,\text{dot}} + C_2 v_r + C_3 u_2 - C_3^2 \tau & \text{(Constraint 4)} \\
0 = v_r \sin\theta + r(\cos\theta)\theta_{\text{dot}} + v_d - v_x & \text{(Constraint 5)}' \\
0 = v_r \cos\theta - r(\sin\theta)\theta_{\text{dot}} - v_z & \text{(Constraint 6)}' \\
0 = p_1'(t) - v_x & \text{(Constraint 7)}' \\
0 = p_2'(t) - v_z & \text{(Constraint 8)}'
\end{cases}$$

$$\begin{cases}
0 = r \sin\theta + d - x & \text{(Constraint 5)} \\
0 = r \cos\theta - z & \text{(Constraint 6)} \\
0 = p_1(t) - x & \text{(Constraint 7)} \\
0 = p_2(t) - z & \text{(Constraint 8)}
\end{cases}$$

The red algebraic equations are the new algebraic equation after the derivation. The green algebraic equation is the hidden constraints that is removed from the DAE system but that is satisfied by choosing the right initial conditions.

The red part cannot determine the algebraic variable and must be differentiated again.

Step 2: Differentiate (again) the algebraic constraints

To find the right candidate(s) here we write a table of the dependencies

	$v_{x,\text{dot}}$	$v_{z,\text{dot}}$	$v_{d,\text{dot}}$	$v_{r,\text{dot}}$	u_1	u_2	τ	θ_{dot}
(Constraint 1)	1	0	0	0	0	0	1	0
(Constraint 2)	0	1	0	0	0	0	1	0
(Constraint 3)	0	0	1	0	1	0	1	0
(Constraint 4)	0	0	0	1	0	1	1	0
(Constraint 5)′	0	0	0	0	0	0	0	1
(Constraint 6)′	0	0	0	0	0	0	0	1
(Constraint 7)′	0	0	0	0	0	0	0	0
(Constraint 8)′	0	0	0	0	0	0	0	0

The last two constraints are candidate for differentiation as it doesn't introduce new

algebraic variables. Differentiate thus the last constraints and substitute known variable in differential form using the ODE part

$$\begin{aligned} &\text{(Constraint 7)}'' \quad 0 = p_1''(t) - v_{x,\text{dot}} \\ &\text{(Constraint 8)}'' \quad 0 = p_2''(t) - v_{z,\text{dot}} \end{aligned}$$

and constraint part of the DAE become

$$\begin{cases} 0 = M_2 v_{x,\text{dot}} + \tau \sin\theta & \text{(Constraint 1)} \\ 0 = M_2 v_{z,\text{dot}} + \tau \cos\theta - mg & \text{(Constraint 2)} \\ 0 = M_1 v_{d,\text{dot}} + C_1 v_d - u_1 - \tau \sin\theta & \text{(Constraint 3)} \\ 0 = J v_{r,\text{dot}} + C_2 v_r + C_3 u_2 - C_3^2 \tau & \text{(Constraint 4)} \\ 0 = v_r \sin\theta + r(\cos\theta)\theta_{\text{dot}} + v_d - v_x & \text{(Constraint 5)}' \\ 0 = v_r \cos\theta - r(\sin\theta)\theta_{\text{dot}} - v_z & \text{(Constraint 6)}' \\ 0 = p_1''(t) - v_{x,\text{dot}} & \text{(Constraint 7)}'' \\ 0 = p_2''(t) - v_{z,\text{dot}} & \text{(Constraint 8)}'' \end{cases}$$

$$\begin{cases} 0 = r\sin\theta + d - x & \text{(Constraint 5)} \\ 0 = r\cos\theta - z & \text{(Constraint 6)} \\ 0 = p_1(t) - x & \text{(Constraint 7)} \\ 0 = p_2(t) - z & \text{(Constraint 8)} \\ 0 = p_1'(t) - v_x & \text{(Constraint 7)}' \\ 0 = p_2'(t) - v_z & \text{(Constraint 8)}' \end{cases}$$

Step 3: Differentiate (again) the algebraic constraints

To find the right candidate(s) here we write a table of the dependencies

	$v_{x,\text{dot}}$	$v_{z,\text{dot}}$	$v_{d,\text{dot}}$	$v_{r,\text{dot}}$	u_1	u_2	τ	θ_{dot}
(Constraint 1)	1	0	0	0	0	0	1	0
(Constraint 2)	0	1	0	0	0	0	1	0
(Constraint 3)	0	0	1	0	1	0	1	0
(Constraint 4)	0	0	0	1	0	1	1	0
(Constraint 5)′	0	0	0	0	0	0	0	1
(Constraint 6)′	0	0	0	0	0	0	0	1
(Constraint 7)″	1	0	0	0	0	0	0	0
(Constraint 8)″	0	1	0	0	0	0	0	0

There are no constraints candidate for derivation. We can check if the constraints are enough to determine algebraic variable.

We use sprank of MATLAB to check if the matrix can be non singular.

```
sparsity_pattern = [...
1, 0, 0, 0, 0, 0, 1, 0; ...
0, 1, 0, 0, 0, 0, 1, 0; ...
0, 0, 1, 0, 1, 0, 1, 0; ...
0, 0, 0, 1, 0, 1, 1, 0; ...
0, 0, 0, 0, 0, 0, 0, 1; ...
0, 0, 0, 0, 0, 0, 0, 1; ...
1, 0, 0, 0, 0, 0, 0, 0; ...
0, 1, 0, 0, 0, 0, 0, 0; ...
];

fprintf('Estimated rank = %d\n', sprank(sparsity_pattern) );

> Estimated rank = 6
```

Matrix is not full rank. To continue the algorithm we can apply elimination to remove redundancy. First of all rewrite the constraint as a linear system

$$\underbrace{\begin{bmatrix} M_2 & 0 & 0 & 0 & 0 & 0 & \sin(\theta) & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 & \cos(\theta) & 0 \\ 0 & 0 & M_1 & 0 & -1 & 0 & -\sin(\theta) & 0 \\ 0 & 0 & 0 & J & 0 & C_3 & -C_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r\cos(\theta) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r\sin(\theta) \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{pmatrix} v_{x,\mathrm{dot}} \\ v_{z,\mathrm{dot}} \\ v_{d,\mathrm{dot}} \\ v_{r,\mathrm{dot}} \\ u_1 \\ u_2 \\ \tau \\ \theta_{\mathrm{dot}} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 \\ mg \\ -C_1 v_d \\ -C_2 v_r \\ v_x - v_d - v_r \sin(\theta) \\ v_z - v_r \cos(\theta) \\ -p_1''(t) \\ -p_2''(t) \end{bmatrix}}_{b}$$

given $\boldsymbol{L}_1$ defined forward we have

$$\boldsymbol{L}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ M_2^{-1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & M_2^{-1} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{L}_1 \boldsymbol{A} = \begin{bmatrix} M_2 & 0 & 0 & 0 & 0 & 0 & \sin(\theta) & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 & \cos(\theta) & 0 \\ 0 & 0 & M_1 & 0 & -1 & 0 & -\sin(\theta) & 0 \\ 0 & 0 & 0 & J & 0 & C_3 & -C_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r\cos(\theta) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r\sin(\theta) \\ 0 & 0 & 0 & 0 & 0 & 0 & M_2^{-1}\sin(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_2^{-1}\cos(\theta) & 0 \end{bmatrix}$$

given $\boldsymbol{R}_1$ defined forward we have

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\boldsymbol{R}_1\boldsymbol{L}_1\boldsymbol{A} = \begin{bmatrix} M_2 & 0 & 0 & 0 & 0 & 0 & \sin(\theta) & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 & \cos(\theta) & 0 \\ 0 & 0 & M_1 & 0 & -1 & 0 & -\sin(\theta) & 0 \\ 0 & 0 & 0 & J & 0 & C_3 & -C_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_2^{-1} & 0 \end{bmatrix}$$

$\boldsymbol{R}_1\boldsymbol{L}_1\boldsymbol{A}$ have two rows zero, thus multiplying by $\boldsymbol{R}_1\boldsymbol{L}_1$ the constraints permits to find two algebraic constraints that can be derived.

$$\boldsymbol{R}_1\boldsymbol{L}_1 \begin{bmatrix} M_2 v_{x,\text{dot}} + \tau \sin\theta \\ M_2 v_{z,\text{dot}} + \tau\cos\theta - mg \\ M_1 v_{d,\text{dot}} + C_1 v_d - u_1 - \tau\sin\theta \\ J v_{r,\text{dot}} + C_2 v_r + C_3 u_2 - C_3^2\tau \\ v_r \sin\theta + r(\cos\theta)\theta_{\text{dot}} + v_d - v_x \\ v_r\cos\theta - r(\sin\theta)\theta_{\text{dot}} - v_z \\ p_1''(t) - v_{x,\text{dot}} \\ p_2''(t) - v_{z,\text{dot}} \end{bmatrix} = \begin{bmatrix} M_2 v_{x,\text{dot}} + \tau\sin(\theta) \\ M_2 v_{z,\text{dot}} + \tau\cos(\theta) - mg \\ M_1 v_{d,\text{dot}} + C_1 v_d - u_1 - \tau\sin(\theta) \\ J v_{r,\text{dot}} + C_2 v_r + C_3 u_2 - C_3^2\tau \\ v_d - v_x + \cos(\theta) + r\theta_{\text{dot}} + \sin(\theta)v_z \\ (v_d - v_x)\sin(\theta) - \cos(\theta)v_z + v_r \\ \cos(\theta)p_1''(t) - p_2''(t)\sin(\theta) + M_2^{-1}\sin(\theta)mg \\ \cos(\theta)p_2''(t) + p_1''(t)\sin(\theta) + M_2^{-1}(\tau - \cos(\theta))mg \end{bmatrix}$$

Collecting the recombined constraints

$$\begin{cases} 0 = M_2 v_{x,\text{dot}} + \tau\sin\theta & \text{(C1)} \\ 0 = M_2 v_{z,\text{dot}} + \tau\cos(\theta) - mg & \text{(C2)} \\ 0 = M_1 v_{d,\text{dot}} + C_1 v_d - u_1 - \tau\sin(\theta) & \text{(C3)} \\ 0 = J v_{r,\text{dot}} + C_2 v_r + C_3 u_2 - C_3^2\tau & \text{(C4)} \\ 0 = v_d - v_x + \cos(\theta) + r\theta_{\text{dot}} + \sin(\theta)v_z & \text{(C5)} \\ 0 = (v_d - v_x)\sin(\theta) - \cos(\theta)v_z + v_r & \text{(C6)} \\ 0 = \cos(\theta)p_1''(t) - p_2''(t)\sin(\theta) + M_2^{-1}\sin(\theta)mg & \text{(C7)} \\ 0 = \cos(\theta)p_2''(t) + p_1''(t)\sin(\theta) + M_2^{-1}(\tau - \cos(\theta))mg & \text{(C8)} \end{cases}$$

where we have renamed constraints because now are mixed by the multiplication of $\boldsymbol{L}^{-1}$. The constraints (C6) and (C7) can be derived.

$$\text{(C6)}' = ((v_d - v_x)\theta_{\text{dot}} - v_{z,\text{dot}})\cos(\theta) + (\theta_{\text{dot}}v_z + v_{d,\text{dot}} - v_{x,\text{dot}})\sin(\theta) + v_{r,\text{dot}}$$

$$\text{(C7)}' = p_1'''(t)\cos(\theta) - p_2'''(t)\sin(\theta) - \left(p_2''(t)\cos(\theta) + p_1''(t)\sin(\theta)\right)\theta_{\text{dot}} + M_2^{-1}\theta_{\text{dot}}\cos(\theta)mg$$

and the constraints and hidden constraints become:

$$\begin{cases} 0 = M_2 v_{x,\mathrm{dot}} + \tau \sin\theta & \text{(C1)} \\ 0 = M_2 v_{z,\mathrm{dot}} + \tau \cos(\theta) - mg & \text{(C2)} \\ 0 = M_1 v_{d,\mathrm{dot}} + C_1 v_d - u_1 - \tau \sin(\theta) & \text{(C3)} \\ 0 = J v_{r,\mathrm{dot}} + C_2 v_r + C_3 u_2 - C_3^2 \tau & \text{(C4)} \\ 0 = v_d - v_x + \cos(\theta) + r\theta_{\mathrm{dot}} + \sin(\theta) v_z & \text{(C5)} \\ 0 = ((v_d - v_x)\theta_{\mathrm{dot}} - v_{z,\mathrm{dot}})\cos(\theta) + (\theta_{\mathrm{dot}} v_z + v_{d,\mathrm{dot}} - v_{x,\mathrm{dot}})\sin(\theta) + v_{r,\mathrm{dot}} & \text{(C6)}' \\ 0 = p_1'''(t)\cos(\theta) - p_2'''(t)\sin(\theta) - \left(p_2''(t)\cos(\theta) + p_1''(t)\sin(\theta)\right)\theta_{\mathrm{dot}} + M_2^{-1}\theta_{\mathrm{dot}}\cos(\theta)mg & \text{(C7)}' \\ 0 = \cos(\theta)p_2''(t) + p_1''(t)\sin(\theta) + M_2^{-1}(\tau - \cos(\theta)mg & \text{(C8)} \end{cases}$$

$$\begin{cases} 0 = r\sin\theta + d - x & \text{(Constraint 5)} \\ 0 = r\cos\theta - z & \text{(Constraint 6)} \\ 0 = p_1(t) - x & \text{(Constraint 7)} \\ 0 = p_2(t) - z & \text{(Constraint 8)} \\ 0 = p_1'(t) - v_x & \text{(Constraint 7)}' \\ 0 = p_2'(t) - v_z & \text{(Constraint 8)}' \\ 0 = (v_d - v_x)\sin(\theta) - \cos(\theta)v_z + v_r & \text{(C6)} \\ 0 = \cos(\theta)p_1''(t) - p_2''(t)\sin(\theta) + M_2^{-1}\sin(\theta)mg & \text{(C7)} \end{cases}$$

Step 4: Differentiate the algebraic constraints?

To find the right candidate(s) here we write a table of the dependencies

	$v_{x,\mathrm{dot}}$	$v_{z,\mathrm{dot}}$	$v_{d,\mathrm{dot}}$	$v_{r,\mathrm{dot}}$	u_1	u_2	τ	θ_{dot}
(C1)	1	0	0	0	0	0	1	0
(C2)	0	1	0	0	0	0	1	0
(C3)	0	0	1	0	1	0	1	0
(C4)	0	0	0	1	0	1	1	0
(C5)	0	0	0	0	0	0	0	1
(C6)′	1	1	1	1	0	0	0	1
(C7)′	0	0	0	0	0	0	0	1
(C8)	0	0	0	0	0	0	1	0

There are no constraints candidate for derivation. We can check if the constraints are enough to determine algebraic variable.

We use sprank of MATLAB to check if the matrix can be non singular.

```
sparsity_pattern = [...
1, 0, 0, 0, 0, 0, 1, 0;...
0, 1, 0, 0, 0, 0, 1, 0;...
0, 0, 1, 0, 1, 0, 1, 0;...
0, 0, 0, 1, 0, 1, 1, 0;...
0, 0, 0, 0, 0, 0, 0, 1;...
```

```
1, 1, 1, 1, 0, 0, 0, 1;...
0, 0, 0, 0, 0, 0, 0, 1;...
0, 0, 0, 0, 0, 0, 1, 0;...
];

fprintf('Estimated rank = %d\n', sprank(sparsity_pattern) );

> Estimated rank = 7
```

the matrix is singular. Instead of apply LU try to combine (C5) and (C7)$'$ to eliminate θ_{dot}

$$\begin{aligned}\widetilde{\text{C7}} &= \text{(C5)}\left(M_2^{-1}mg\cos(\theta) - p_2''(t)\cos(\theta) - p_1''(t)\sin(\theta)\right) + \text{(C7)}'r \\ &= \left(\cos(\theta)p_1'(t) - \cos(\theta)v_d - \sin(\theta)p_2'(t)\right)\left(\cos(\theta)p_2''(t) + \sin(\theta)p_1''(t)\right) \\ &+ \left(\sin(\theta)p_2'''(t) - \cos(\theta)p_1'''(t)\right)r \\ &+ M_2^{-1}\left(\cos(\theta)v_d - \cos(\theta)p_1'(t) + \sin(\theta)p_2'(t)\right)\cos(\theta)mg\end{aligned}$$

and

$$\begin{aligned}\widetilde{\text{C7}}' &= \left(M_2^{-1}\left(2p_2'\theta_{\text{dot}} - p_1'' + v_{d,\text{dot}}\right)mg + \left(p_1' - v_d\right)p_2''' + p_1''p_2'' + 2\theta_{\text{dot}}(p_1' - v_d)p_1'' - \left(2\theta_{\text{dot}}p_2' + v_{d,\text{dot}}\right)p_2''\right)\cos \\ &+ \left(\left(M_2^{-1}\left(p_2''(t) + 2\theta_{\text{dot}}(p_1'(t) - v_d)\right)mg + (p_1' - v_d)p_1''' - p_2'''p_2' + p_1''p_1'' - p_2''p_2'' + \left(-2\theta_{\text{dot}}p_2' - v_{d,\text{dot}}\right)p_1''\right.\right. \\ &+ \left(-p_2'p_1''' - p_2''p_1''\right)\sin(\theta)^2 + \\ &\left(r\theta_{\text{dot}}p_1''' + rp_2'''' + v_rp_2'''\right)\sin(\theta) - M_2^{-1}p_2'\theta_{\text{dot}}mg + \left(-\theta_{\text{dot}}p_1' + v_d\theta_{\text{dot}}\right)p_1'' + p_2''\theta_{\text{dot}}p_2'\end{aligned}$$

TO BE COMPLETED

3. DAE Exercises

The exercises are marked **Fundamental**, **Computational**, or **Advanced**. Hints indicate a starting point without reproducing a solution.

3.1 Fundamental

1. For $x' = z$ and $0 = x^2 + z^2 - 1$, determine all consistent values of $z(0)$ for a prescribed $x(0)$. Which choices cease to be regular? *Hint:* inspect $\partial g/\partial z$.
2. Starting from a holonomic constraint $\Phi(\boldsymbol{q}) = 0$, derive the velocity and acceleration constraints and identify the terms that contain the multiplier. *Hint:* differentiate with the chain rule before substituting the equations of motion.
3. Explain why enforcing only the acceleration constraint permits position drift. *Hint:* write the residual dynamics and count integrations.

3.2 Computational

1. Implement the consistent projection of the course notes for the redundant-coordinate pendulum. Compare the correction obtained with two different diagonal scalings $\boldsymbol{W}$. *Hint:* report both scaled and physical residuals.
2. Integrate the same pendulum without stabilization, with Baumgarte stabilization, and with projection. Compare position drift, velocity drift, and energy error. *Hint:* use identical time steps and nonlinear tolerances.
3. Perform a three-level refinement study for one differential variable and one algebraic variable. *Hint:* estimate the two observed orders separately, using Order Reduction in DAE Integration of the course notes.

3.3 Advanced

1. For an index-2 Hessenberg DAE, derive conditions under which the consistency Jacobian is nonsingular. *Hint:* use the implicit-function theorem on the differentiated algebraic constraint.
2. Compare differential, perturbation, tractability, and strangeness indices for a constant-coefficient regular pencil. *Hint:* first transform the pencil to Weierstrass form and state the regularity assumptions.
3. Design an event restart for a mechanical system that acquires a new constraint at impact. *Hint:* separate the impulsive velocity projection from the post-event consistent initialization.

III Calculus of Variations

4 Worked Exercises 89
4.1 Exercise 1 — Fundamental
4.2 Exercise 2 — Computational
4.3 Exercise 3 — Advanced

Worked exercises in the calculus of variations. The corresponding theory is in the part of the same name of the course notes.

4. Worked Exercises

The three worked problems progress from **fundamental** to **computational** and **advanced**. In each case, start by identifying admissible variations, perform integration by parts only after computing the first variation, and separate interior equations from boundary terms.

4.1 Exercise 1 — Fundamental

Find stationary point of the functional

$$\mathcal{F}(x) = \int_0^2 \left(x(t)^2 + x(t)(x'(t))^2 + tx(t)\right)\,\mathrm{d}t$$

with constraint

$$x(0) = 0, \qquad x(2) = 1$$

4.1.1 First Variation

We start by applying the first variation to the functional: $\mathcal{F}(x)$. To compute the variation, we first take the variation of the integrand:

$$\begin{aligned}
\delta\mathcal{F}(x) &= \delta \int_0^2 \left(x(t)^2 + x(t)(x'(t))^2 + tx(t)\right)\,\mathrm{d}t \\
&= \int_0^2 \delta\left(x(t)^2 + x(t)(x'(t))^2 + tx(t)\right)\,\mathrm{d}t \\
&= \int_0^2 \left(2x(t)\delta x(t) + \delta x(t)(x'(t))^2 + 2x(t)x'(t)\delta x'(t) + t\delta x(t)\right)\,\mathrm{d}t \\
&= \int_0^2 \left(\left(2x(t) + (x'(t))^2 + t\right)\delta x(t) + 2x(t)x'(t)\delta x'(t)\right)\,\mathrm{d}t
\end{aligned}$$

Here, we expanded the variation term by term and grouped the expressions. The blue term $2x(t)x'(t)\delta x'(t)$ is crucial for the integration by parts.

4.1.2 Integration by Parts

To eliminate the $\delta x'(t)$ term, we apply integration by parts to the expression $2x(t)x'(t)\delta x'(t)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(2x(t)x'(t)\delta x(t)\right) &= 2x(t)x'(t)\delta x'(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(2x(t)x'(t)\right)\delta x(t)\\ &= 2x(t)x'(t)\delta x'(t) + 2\left((x'(t))^2 + x(t)x''(t)\right)\delta x(t)\end{aligned}$$

This allows us to rewrite the variation of $\mathcal{F}(x)$ as:

$$\begin{aligned}\delta\mathcal{F}(x) &= \int_0^2 \left(2x(t) + (x'(t))^2 + t\right)\delta x(t) + 2x(t)x'(t)\delta x'(t)\,\mathrm{d}t\\ &= \int_0^2 \left(2x(t) + (x'(t))^2 + t\right)\delta x(t)\,\mathrm{d}t\\ &\quad + \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(2x(t)x'(t)\delta x(t)\right) - 2\left((x'(t))^2 + x(t)x''(t)\right)\delta x(t)\right]\mathrm{d}t\end{aligned}$$

Now, after applying the integration by parts, we can group terms again:

$$\begin{aligned}\delta\mathcal{F}(x) &= \int_0^2 \left(2x(t) - (x'(t))^2 + t - 2x(t)x''(t)\right)\delta x(t)\,\mathrm{d}t\\ &\quad + \Big[2x(t)x'(t)\delta x(t)\Big]_0^2\end{aligned}$$

Since $\delta x(0) = \delta x(2) = 0$ (from the boundary conditions), the boundary term vanishes. Therefore, we are left with:

$$\delta\mathcal{F}(x) = \int_0^2 \left(2x(t) - (x'(t))^2 + t - 2x(t)x''(t)\right)\delta x(t)\,\mathrm{d}t$$

4.1.3 Euler–Lagrange Equation

For the variation $\delta\mathcal{F}(x)$ to be zero for all variations $\delta x(t)$, the integrand must vanish. Therefore, we obtain the Euler-Lagrange equation:

$$2x(t) - (x'(t))^2 + t - 2x(t)x''(t) = 0$$

4.1.4 Boundary Conditions

The problem also specifies boundary conditions:

$$x(0) = 0, \quad x(2) = 1$$

4.1.5 Final Problem

Thus, the full problem is to solve the following boundary value problem:

$$\begin{cases} 2x(t) - (x'(t))^2 + t - 2x(t)x''(t) = 0, \\ x(0) = 0, \\ x(2) = 1. \end{cases}$$

4.2 Exercise 2 — Computational

Find stationary points of the functional

$$\mathcal{F}(x) = \int_a^b x^2 + 4xx' + 2(x')^2\,\mathrm{d}t$$

with constraint

$$x(a) = x_a$$

4.2.1 First Variation

$$\mathcal{F}(x+\alpha\delta x) = \int_a^b (x+\alpha\delta x)^2 + 4(x+\alpha\delta x)(x'+\alpha\delta x') + 2(x'+\alpha\delta x')^2 \,\mathrm{d}t$$

perform variation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{F}(x+\alpha\delta x) &= \underbrace{\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b}_{\leftrightarrow} \cdots \,\mathrm{d}t \\ &= \underbrace{\int_a^b \frac{\mathrm{d}}{\mathrm{d}\alpha}}_{\leftrightarrow} \cdots \,\mathrm{d}t \\ &= \int_a^b (x+\alpha\delta x)(2\delta x + 4\delta x') + 4(x'+\alpha\delta x')(\delta x+\delta x')\,\mathrm{d}t\end{aligned}$$

evaluating at $\alpha = 0$:

$$\begin{aligned}\delta\mathcal{F}(x) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{F}(x+\alpha\delta x)\Big|_{\alpha=0} \\ &= \int_a^b x(2\delta x + 4\delta x') + 4x'(\delta x+\delta x')\,\mathrm{d}t \\ &= \int_a^b (2x+4x')\delta x + 4(x+x')\delta x'\,\mathrm{d}t\end{aligned}$$

4.2.2 Integration by Parts

For the term $4(x+x')\delta x'$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left((x+x')\delta x\right) &= (x+x')\delta x' + \frac{\mathrm{d}}{\mathrm{d}t}\left((x+x')\right)\delta x \\ &= (x+x')\delta x' + (x'+x'')\delta x\end{aligned}$$

and

$$\begin{aligned}\int_a^b (x+x')\delta x'\,\mathrm{d}t &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left((x+x')\delta x\right)\,\mathrm{d}t - \int_a^b (x'+x'')\delta x\,\mathrm{d}t \\ &= (x+x')\delta x\Big|_a^b - \int_a^b (x'+x'')\delta x\,\mathrm{d}t\end{aligned}$$

so that

$$\begin{aligned}\delta\mathcal{F}(x) = \int_a^b (2x + \cancel{4x'} - \cancel{4x'} - 4x'')\delta x\,\mathrm{d}t + \\ 4(x(b)+x'(b))\delta x(b) - 4\underbrace{\cancel{(x(a)+x'(a))\delta x(a)}}_{\delta x(a)=0}\end{aligned}$$

4.2.3 Euler–Lagrange Equation

$$\begin{aligned}\delta x(b) &\implies x(b)+x'(b) = 0 \\ \delta x &\implies 2x(t) - 4x''(t) = 0\end{aligned}$$

4.2.4 Final Problem

$$\begin{cases} x - 2x'' = 0 \\ x(a) = x_a \\ x(b) + x'(b) = 0 \end{cases}$$

The same result can be obtained using Euler-Lagrange equation

$$L(x, x', t) = x^2 + 4xx' + 2(x')^2$$

so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L(x, x', t)}{\partial x'} - \frac{\partial L(x, x', t)}{\partial x} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}(4x + 4x') - (2x + 4x') &= 0 \\ 4x'' - 2x &= 0 \\ -2(x - 2x'') &= 0 \end{aligned}$$

and for the free boundary

$$0 = \left.\frac{\partial L(x, x', t)}{\partial x'}\right|_{t=b} = \left.(4x + 4x')\right|_{t=b} = 4(x(b) + x'(b)) = 0$$

4.3 Exercise 3 — Advanced

Find stationary points of the functional, assuming $x'(t) > 0$:

$$\mathcal{F}(x) = \int_0^1 x'(\log x')^2 \mathrm{d}t$$

with free endpoint values.

4.3.1 First Variation

$$\begin{aligned} \delta\mathcal{F}(x) &= \delta \int_0^1 x'(\log x')^2 \mathrm{d}t \\ &= \int_0^1 \delta\left(x'(\log x')^2\right) \mathrm{d}t \\ &= \int_0^1 \delta x'(\log x')^2 + 2x' \log x' \frac{\delta x'}{x'} \mathrm{d}t \\ &= \int_0^1 \left((\log x')^2 + 2\log x'\right) \delta x' \mathrm{d}t \end{aligned}$$

R Do not do this wrong argument: δx is arbitrary and so that using fundamental lemma of calculus of variations it follows

$$(\log x')^2 + 2\log x' = 0$$

the lemma is true for δx not $\delta x'$.

4.3.2 Integration by Parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left((\log x')^2 + 2\log x'\right)\delta x\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[(\log x')^2 + 2\log x'\right]\delta x + \left((\log x')^2 + 2\log x'\right)\delta x'$$

so that

$$\delta\mathcal{F}(x) = \left((\log x')^2 + 2\log x'\right)\delta x\Big|_0^1 - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left[(\log x')^2 + 2\log x'\right]\delta x\,\mathrm{d}t$$

4.3.3 Euler–Lagrange Equation

$$\begin{aligned}
\delta x(a) &\implies (\log x'(a))^2 + 2\log x'(a) = 0\\
\delta x(b) &\implies (\log x'(b))^2 + 2\log x'(b) = 0\\
\delta x(t) &\implies \frac{\mathrm{d}}{\mathrm{d}t}\left[(\log x')^2 + 2\log x'\right] = 0,
\end{aligned}$$

4.3.4 Final Problem

$$\begin{cases}
(\log x')^2 + 2\log x' = C,\\
(\log x'(a))^2 + 2\log x'(a) = 0\\
(\log x'(b))^2 + 2\log x'(b) = 0
\end{cases}$$

The natural boundary conditions force $C = 0$ throughout the interval. Hence

$$\log x'\left(\log x' + 2\right) = 0,$$

so every smooth stationary function is affine with one of the two admissible slopes:

$$x(t) = c + t \qquad \text{or} \qquad x(t) = c + \mathrm{e}^{-2}t, \qquad c \in \mathbb{R}.$$

IV Optimal Control

5 Examples of Optimal Control Problems .. 99

5.1 The Brachistochrone Problem
5.2 Dido's Isoperimetric Problem
5.3 An Optimal Control Problem with Path Constraints
5.4 Dubins Car
5.5 Four Singular Optimal Control Problems
5.6 Free-Time Problems

Optimal control problems solved in full. The corresponding theory is in the part of the same name of the course notes.

5. Examples of Optimal Control Problems

5.1 The Brachistochrone Problem

One of the earliest and most renowned challenges in the calculus of variations is the brachistochrone problem, originally posed by Johann Bernoulli in 1696. This problem can be approached from various perspectives, particularly through the lenses of calculus of variations and optimal control theory, which involve ordinary and differential algebraic equations (ODEs and DAEs), as well as the principles of energy conservation.

5.1.1 Problem Statement

The goal of the brachistochrone problem is to determine the path of minimum time that connects two points under the influence of gravity alone. Specifically, we consider motion in two dimensions, transitioning from a starting point A to a fixed endpoint B. The challenge is to find the shape of the curve along which a particle, subject solely to gravitational force, travels from A to B in the least amount of time.

To facilitate analysis, we can establish a coordinate system with the origin at point A. In this system, the vertical axis y points upward (opposite to the direction of gravitational force), while the horizontal axis x is perpendicular to y, ensuring that the x-coordinate of point B is positive. Given that the optimal path γ must be a continuous curve without loops, we can represent γ as a graph $y(x)$.

Because the physical vertical coordinate y is positive upward, points below the release point have $y < 0$. It is therefore convenient in the variational derivation to introduce the nonnegative downward depth

$$h(x) = -y(x) \geq 0.$$

Conservation of energy gives

$$mgh = \frac{1}{2}mv^2 \qquad \Longrightarrow \qquad v = \sqrt{2gh}.$$

This relationship indicates that the solution is independent of the mass m.

5.1.2 Path Length and Travel Time

The distance L traveled by the particle as it approaches endpoint B can be expressed using a line integral:

$$L(h, x) = \oint_\gamma \mathrm{d}s = \int_0^{x_B} \sqrt{1 + (h'(x))^2}\,\mathrm{d}x.$$

The time taken to traverse this path is given by:

$$T(h, x) = \int_0^{x_B} \frac{\sqrt{1 + (h'(x))^2}}{\sqrt{2gh(x)}}\,\mathrm{d}x. \tag{5.1}$$

5.1.3 Constraints

The solution to this problem involves minimizing the functional defined in Equation 5.1 under the following constraints:

$$h(0) = 0, \qquad h(x_B) = h_B = -y_B > 0. \tag{5.2}$$

5.1.4 Solution Using Calculus of Variations

To solve the brachistochrone problem using calculus of variations, we employ the Euler-Lagrange equation. Since the functional

$$T(h) = \int F(h, h')\,\mathrm{d}x$$

does not explicitly depend on x, the Euler-Lagrange equation simplifies to Beltrami's identity:

$$\frac{\partial F}{\partial h} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial h'} = 0 \quad \Longrightarrow \quad F - h'\frac{\partial F}{\partial h'} = c, \tag{5.3}$$

where c is a constant determined by the constraints given in Equation 5.2.

For the brachistochrone problem, we can express F as:

$$\sqrt{2g}T = \int_0^{x_B} \sqrt{\frac{1 + h'(x)^2}{h(x)}}\,\mathrm{d}x \quad \Longrightarrow \quad F(h, h') = \sqrt{\frac{1 + h'(x)^2}{h(x)}}.$$

Thus, the Euler-Lagrange equation, Equation 5.3, reduces to:

$$\sqrt{\frac{1 + h'^2}{h}} - \frac{h'^2}{\sqrt{h(1 + h'^2)}} = c \quad \Longrightarrow \quad h(1 + h'^2) = \frac{1}{c^2} = C > 0.$$

This results in an autonomous nonlinear differential equation that can be solved using separation of variables:

$$\int_0^{h_B} \sqrt{\frac{h}{C - h}}\,\mathrm{d}h = \int_{x_A}^{x_B} 1\,\mathrm{d}x.$$

To compute this integral, we use the substitution $h = C\left(\sin\frac{\phi}{2}\right)^2$, which gives us the differential

$$\mathrm{d}h = C \sin\frac{\phi}{2}\cos\frac{\phi}{2}\,\mathrm{d}\phi$$

Thus, we have:

$$\begin{aligned}\int \sqrt{\frac{h}{C-h}}\,\mathrm{d}h &= C\int \sqrt{\frac{\left(\sin\frac{\phi}{2}\right)^2}{1-\left(\sin\frac{\phi}{2}\right)^2}}\sin\frac{\phi}{2}\cos\frac{\phi}{2}\,\mathrm{d}\phi \\ &= C\int\left(\sin\frac{\phi}{2}\right)^2\mathrm{d}\phi \\ &= \frac{C}{2}(\phi-\sin\phi)+k,\end{aligned}$$

where k is the integration constant.

Evaluating this integral at the extrema defined by the constraints in Equation 5.2 allows us to determine the constants C and k. Since the starting point A coincides with the origin, we have $k=0$. The constant C depends on the coordinates of point $B=(x_B, y_B)$. Specifically, we find:

$$\begin{aligned}x_B &= \frac{C}{2}(\phi_B-\sin\phi_B),\\ h_B &= \frac{C}{2}(1-\cos\phi_B).\end{aligned} \tag{5.4}$$

In conclusion, the parametric equations for the brachistochrone are given by:

$$\begin{aligned}x(\phi) &= \frac{C}{2}(\phi-\sin\phi),\\ h(\phi) &= \frac{C}{2}(1-\cos\phi), \qquad y(\phi)=-h(\phi).\end{aligned} \tag{5.5}$$

These equations describe a cycloid curve.

The geometric proof provided by Bernoulli contains spurious solutions, as noted in [4]. While the variational approach presents challenges in proving the existence of the brachistochrone, applying optimal control theory simplifies this task. The existence of a solution can be directly established using the Ascoli-Arzelà theorem.

5.1.5 The brachistochrone as an optimal control problem

This example can be treated as an optimal control problem, the control is the angle ϑ of descent of the particle. Splitting the velocity v in its two components along the xy axes, the problem can be states as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= v\sin\vartheta,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -v\cos\vartheta,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= g\cos\vartheta\end{aligned} \tag{5.6}$$

with border conditions, if T is the time used to travel from A to B,

$$\begin{aligned} x(0) = 0, \quad & y(0) = 0, \quad & v(0) = 0, \\ x(T) = x_B, \quad & y(T) = y_B, \quad & v(T) = v_T, \end{aligned} \tag{5.7}$$

for a free final velocity v_T. The functional to be minimized is T.

5.1.6 A Numerical Example

Consider the brachistochrone problem for a particle released from the origin and traveling to the point $B = (10, -3)$. In the variational formulation, we need to determine the constant C. This can be achieved using Equation 5.4. Solving the nonlinear system for C and ϕ_B yields:

$$\phi_B \approx 4.17, \qquad C \approx 3.97.$$

The solution is illustrated in Figure 5.1.

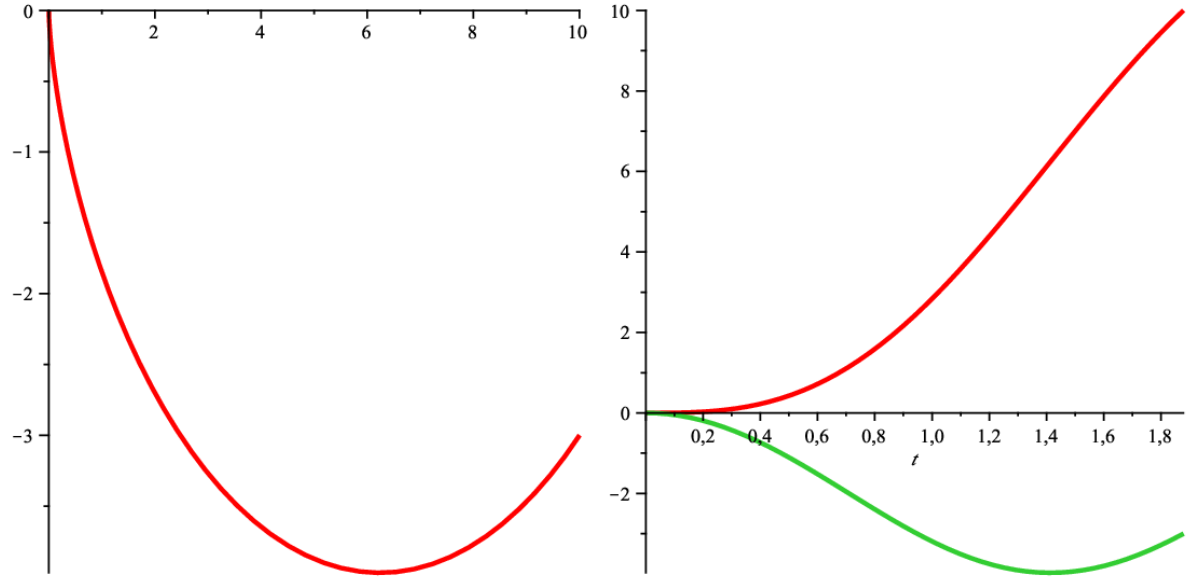


Figure 5.1: **Variational solution to the brachistochrone problem:** The left plot shows the trajectory of the particle from the origin to point B, while the right plot displays the two components of the solution: x in red and y in green.

When treated as an optimal control problem, as stated in Equation 5.6 with boundary conditions Equation 5.7, finding an analytical solution becomes challenging. For instance, the Hamiltonian function is given by:

$$\mathcal{H}(x, y, v, \lambda, \mu, \xi, \theta) = 1 + \lambda v \sin\vartheta - \mu v \cos\vartheta + \xi g \cos\vartheta.$$

According to Pontryagin's Maximum Principle (PMP), the associated boundary value problem is:

$$\begin{aligned} \lambda' &= -\frac{\partial \mathcal{H}}{\partial x} = 0, \\ \mu' &= -\frac{\partial \mathcal{H}}{\partial y} = 0, \\ \xi' &= -\frac{\partial \mathcal{H}}{\partial v} = -\lambda \sin\vartheta + \mu \cos\vartheta. \end{aligned} \tag{5.8}$$

Additionally, we have:

$$0 = \frac{\partial \mathcal{H}}{\partial \vartheta} = \lambda v \cos\vartheta + \mu v \sin\vartheta - \xi g \sin\vartheta.$$

Because the terminal velocity is free, transversality gives $\xi(T) = 0$. Since the terminal time is also free and the problem is autonomous, a normal extremal additionally satisfies $\mathcal{H}(T) = 0$ (and therefore $\mathcal{H}(t) = 0$ along the extremal).

To avoid solving this analytically, we can use the variational formulation's results to inform our optimal control problem (OCP) solution. However, since equations Equation 5.5 are functions of the spatial angle ϕ rather than time t, we need to establish a relationship between them using the time functional:

$$T(x, y) = \int_0^{\phi_B} \frac{\sqrt{x'(\phi)^2 + y'(\phi)^2}}{\sqrt{-2gy(\phi)}}\,\mathrm{d}\phi. \tag{5.9}$$

A brief calculation shows that:

$$T(x, y) = \int_0^{\phi_B} \sqrt{\frac{C}{2g}}\,\mathrm{d}\phi = \phi_B\sqrt{\frac{C}{2g}} = k\phi_B \approx 1.87,$$

which gives an explicit time parametrization of this cycloidal arc. Thus, we can parameterize time as $t = k\phi$, for $t \in [0, k\phi_B]$.

From this perspective, the control $\vartheta(t)$ must satisfy:

$$\begin{aligned}\vartheta(t) &= \arctan\left(\frac{y'(t)}{x'(t)}\right) + \frac{\pi}{2} \\ &= \arctan\left(-\frac{\sin(t/k)}{1-\cos(t/k)}\right) + \frac{\pi}{2}.\end{aligned}$$

This leads to:

$$\begin{aligned}\sin\vartheta &= \frac{1-\cos(t/k)}{\sqrt{2}\sqrt{1-\cos(t/k)}}, \\ \cos\vartheta &= \frac{\sin(t/k)}{\sqrt{2}\sqrt{1-\cos(t/k)}}.\end{aligned}$$

Consequently, the velocity becomes:

$$v(t) = \frac{\sqrt{2}C}{2k}\sqrt{1-\cos(t/k)}.$$

The plots of velocity $v(t)$ and control $\vartheta(t)$ are shown in Figure 5.2.

The differential equations for the spatial components become:

$$\begin{aligned}x'(t) &= v(t)\sin\vartheta = \frac{C}{2k}(1-\cos(t/k)), \\ y'(t) &= -v(t)\cos\vartheta = -\frac{C}{2k}\sin(t/k).\end{aligned}$$

To compute the multipliers λ, μ, and ξ, we can use the Hamiltonian function. By substituting for ξ, we are left with a single function in one unknown, λ. This gives us $\lambda = -k/C$. Thus, we find that the three multipliers are:

$$\begin{aligned}\lambda &= -\frac{k}{C} \approx -0.11, \\ \mu &= -\frac{\lambda\sin(T/k)}{1-\cos(T/k)} \approx -0.06, \\ \xi(t) &= \frac{\sqrt{2}C}{2gk}\frac{\lambda\sin(t/k) + \mu(1-\cos(t/k))}{\sqrt{1-\cos(t/k)}}.\end{aligned}$$

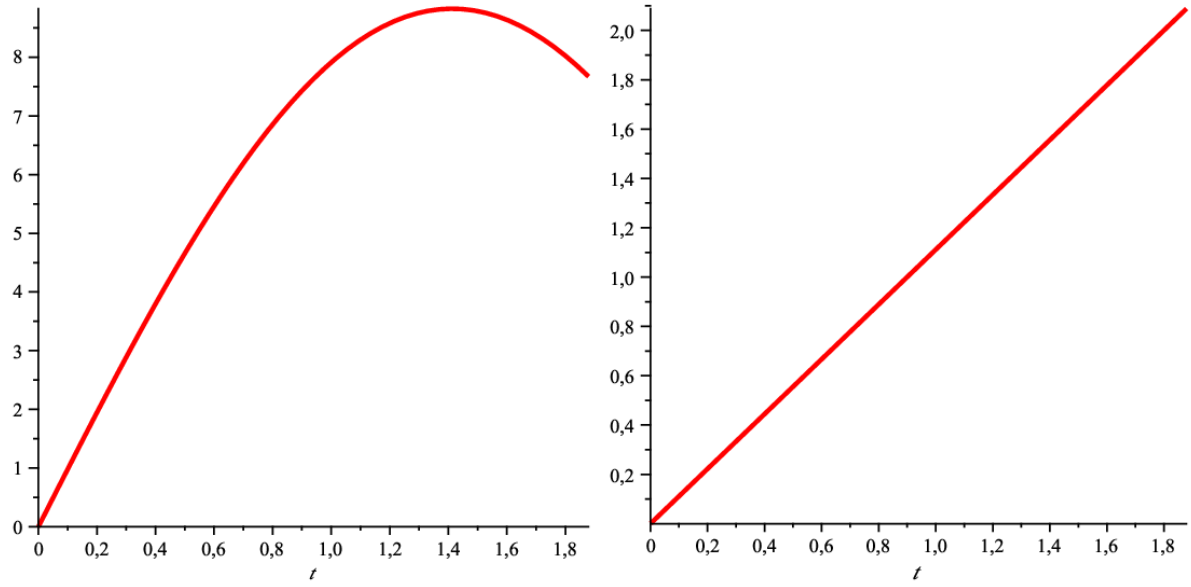


Figure 5.2: On the left is the plot of velocity $v(t)$, and on the right is the control variable $\vartheta(t)$.

The plot of these costate variables is shown in Figure 5.3.

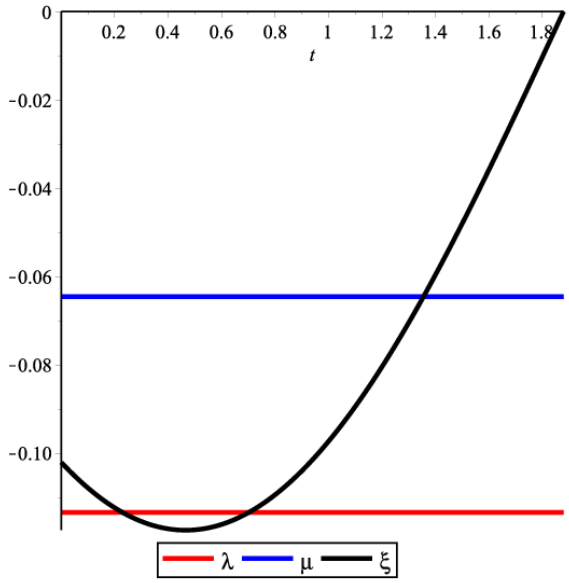


Figure 5.3: The three multipliers are represented as follows: λ in red, μ in blue, and ξ in black.

Finally, using the ACADO toolkit to solve this same problem yields a solution depicted in Figure 5.4.

5.2 Dido's Isoperimetric Problem

Consider a 2D curve defined parametrically, with the following conditions:

- The curve begins and ends on the x-axis.
- The total length of the curve is fixed.

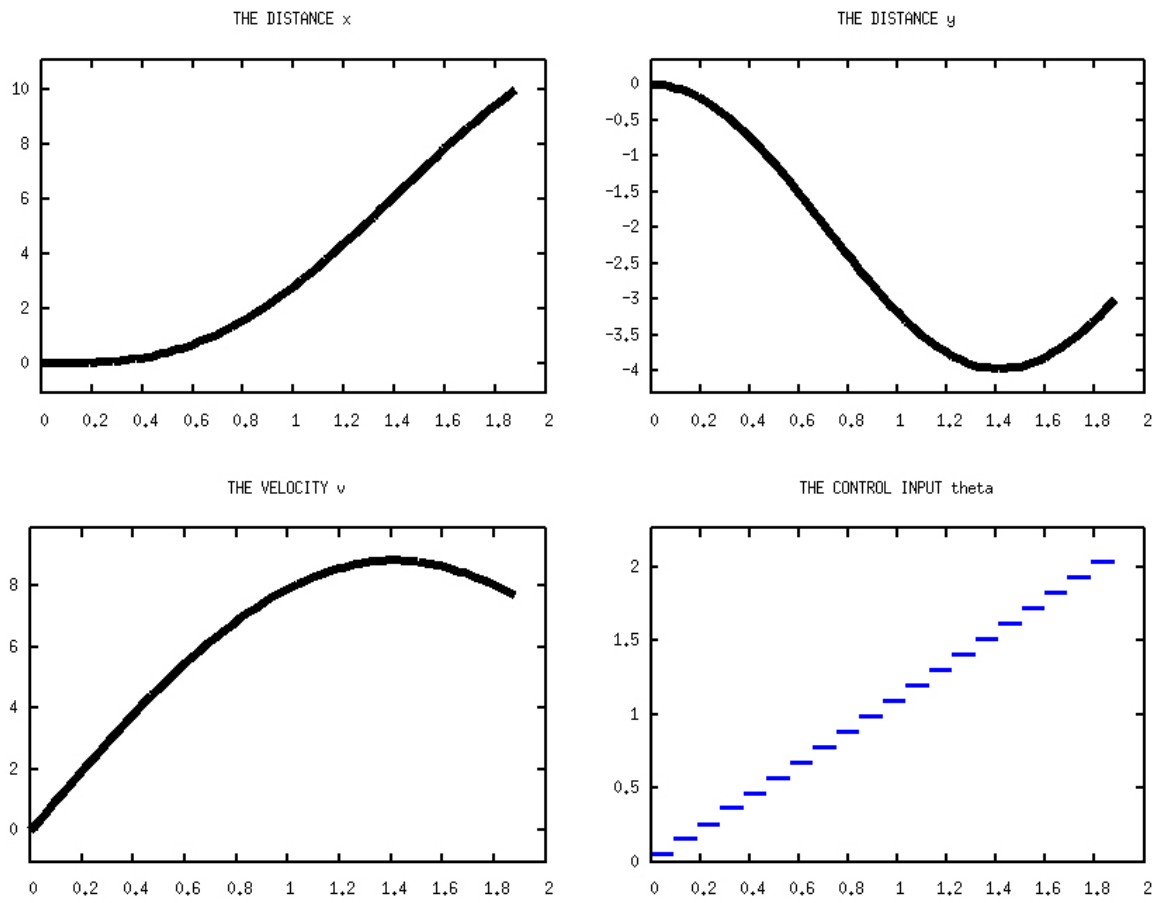


Figure 5.4: **Solution provided by ACADO.**

The objective is to **maximize the area** enclosed between the curve and the x-axis, while maintaining the constraint that the curve's length remains constant.

Formally, we seek the shape of the curve that maximizes the enclosed area, given that:

1. The curve starts and ends on the x-axis.
2. The total length of the curve is fixed.

This problem is often referred to as the *DIDO problem*, and it is a well-known problem in **variational calculus** and **optimal control theory**.

To compute the area inside a curve we require the Divergence Theorem:

5.2.1 The Divergence Theorem

The **Divergence Theorem** is a fundamental result in vector calculus, connecting the flow of a vector field through a closed surface to the behavior of the field within the volume enclosed by the surface. It provides a powerful tool for converting complex surface integrals into simpler volume integrals, and vice versa.

Statement of the Divergence Theorem (2D version)

Let S be a solid region in 2-dimensional space with a smooth (or piecewise smooth), closed boundary curve $C = \partial S$, oriented outward. If $\boldsymbol{F}$ is a continuously differentiable vector field defined on S, the Divergence Theorem states:

$$\int_C \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s = \iint_S (\nabla \cdot \boldsymbol{F}) \, \mathrm{d}S,$$

where:

- $\boldsymbol{F}$: The vector field (e.g., $\boldsymbol{F}(x, y)$).
- $\boldsymbol{n}$: The unit normal vector to the curve C, oriented outward.
- $\nabla \cdot \boldsymbol{F}$: The divergence of $\boldsymbol{F}$, defined as:

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}.$$

2D schema of divergence theorem. Vector field, curve 2D border and normals.

Application of the Divergence Theorem for area computation

Consider a simple two-dimensional vector field:

$$\boldsymbol{F}(x, y) = \alpha x \, \boldsymbol{e}_1 + (1 - \alpha) y \, \boldsymbol{e}_2$$

In this case, the vector field is a linear combination of x and y coordinates, where α is a constant that determines the relative contribution of the x-component and the y-component of the vector field.

5.2.1.1 Divergence of the Vector Field

The divergence of a vector field is the sum of the partial derivatives of its components. For the vector field $\boldsymbol{F}(x, y)$, we compute:

$$\nabla \cdot \boldsymbol{F}(x, y) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$

Substituting the components of $\boldsymbol{F}(x, y)$:

$$\nabla \cdot \boldsymbol{F}(x, y) = \frac{\partial (\alpha x)}{\partial x} + \frac{\partial ((1 - \alpha) y)}{\partial y}$$

Taking the derivatives:

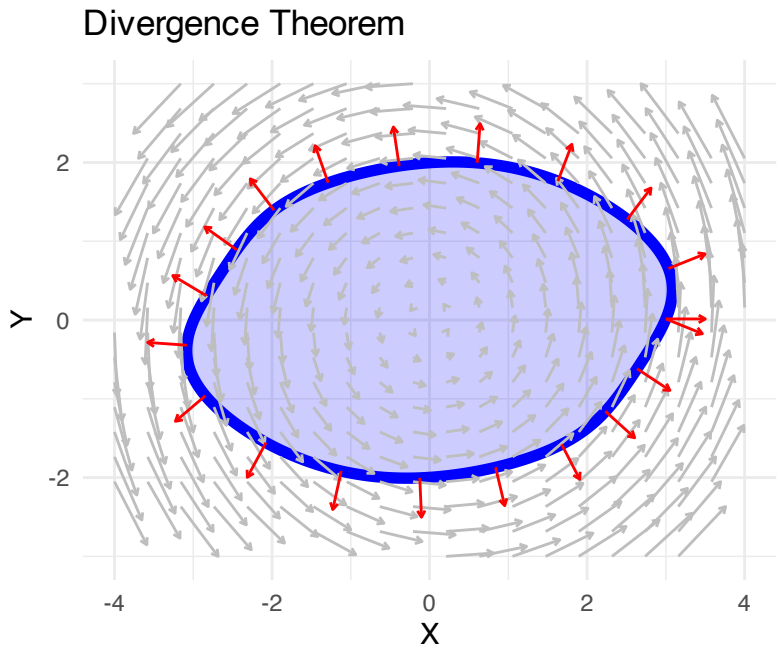


Figure 5.5: Planar illustration of the divergence theorem with outward normals.

$$\nabla \cdot \boldsymbol{F}(x, y) = \alpha + (1 - \alpha) = 1$$

Thus, the divergence of $\boldsymbol{F}(x, y)$ is 1 everywhere in the plane.

Surface Integral and the Divergence Theorem

Now, we apply the **Divergence Theorem** to compute the flux of the vector field through a closed curve (C). The curve is parametrized by:

$$C : (x(t), y(t)), \quad t \in [0, L]$$

The area enclosed by this curve can be computed as a surface integral of the divergence of the vector field over the region enclosed by C. Using the Divergence Theorem, the flux through the closed curve is:

$$|S| = \iint_S 1 \, \mathrm{d}S$$

This is the same as integrating the divergence $\nabla \cdot \boldsymbol{F} = 1$ over the surface, which reduces to the boundary integral:

$$|S| = \int_0^L \boldsymbol{F}(x(t), y(t))^\mathsf{T} \boldsymbol{n}(t) \sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t,$$

In this expression:

- The first term is the **vector field** $\boldsymbol{F}(x(t), y(t))$.
- $\boldsymbol{n}(t) = \big(y'(t), -x'(t)\big)^\mathsf{T} / \sqrt{x'(t)^2 + y'(t)^2}$ is the unit normal to the curve.
- The third term is the **length element** of the curve, $\sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t$.

Expanding the integral:

$$|S| = \int_0^L \big(\alpha x(t) y'(t) - (1 - \alpha) y(t) x'(t)\big) \, \mathrm{d}t$$

Simplifying for a Specific Curve

Suppose we have a curve that passes through the x-axis (i.e., $y(t) = 0$ for some segment of the curve). On this segment, $y'(t) = 0$. In this case, the expression simplifies, and if we choose $\alpha = 1$, the integral becomes:

$$|S| = \int_0^L x(t)y'(t)\,\mathrm{d}t$$

This formula computes the area enclosed by the curve, with the integral focusing on the (x)-coordinate's contribution when the curve intersects the (x)-axis.

Final Expression for the Area

Finally, if the curve is closed and the initial and final points on the curve satisfy $y(0) = y(L) = 0$, the total enclosed area $|S|$ is given by:

$$|S| = \pm \int_0^L x(t)y'(t)\,\mathrm{d}t$$

The sign of the area depends on the orientation of the curve:

- **Positive** if the curve is counterclockwise (the normal points outward).
- **Negative** if the curve is clockwise (the normal points inward).

This sign convention reflects the direction of the normal vector and is crucial for consistent computation of area in physics and engineering applications.

Optimal-Control Formulation of Dido's Problem

Now we are ready to formulate the Dido problem

Minimize

$$s \int_0^L x \sin\theta\,\mathrm{d}t, \quad s = \operatorname{sign}(b - a)$$

Under the constraints

$$x' = \cos\theta$$
$$y' = \sin\theta$$

And boundary conditions

$$x(0) = b, \quad y(0) = 0, \quad x(L) = a, \quad y(L) = 0$$

Solution of Dido's Problem

We start by defining two key functions:

$$H(x, y, \lambda_1, \lambda_2, \theta) = \lambda_1 \cos\theta + (sx + \lambda_2)\sin\theta,$$
$$B(x_0, y_0, x_1, y_1, \omega_1, \omega_2, \omega_3, \omega_4) = \omega_1(x_0 - b) + \omega_2(y_0) + \omega_3(x_1 - a) + \omega_4(y_1),$$

where H represents the Hamiltonian function and B encodes boundary conditions. These functions are used to compute the adjoint equations and boundary conditions for the control problem.

Deriving the Adjoint Equations

From the Hamiltonian H, the adjoint equations are given by:

$$\lambda_1' + H_x = \lambda_1' + s\sin\theta = 0,$$
$$\lambda_2' + H_y = \lambda_2' = 0,$$

which simplify to:

1. $\lambda_2(t) = \text{constant} = -sc_y$,
2. $\lambda_1' = -s\sin\theta = -sy'$,

yielding:

$$\lambda_1(t) = s(c_y - y(t)).$$

Boundary Conditions

The adjoint boundary conditions are derived from the B-function:

$$\begin{aligned} B_{x_0} + \lambda_1(0) &= \omega_1 + \lambda_1(0) = 0, \\ B_{y_0} + \lambda_2(0) &= \omega_2 + \lambda_2(0) = 0, \\ B_{x_1} - \lambda_1(L) &= \omega_3 - \lambda_1(L) = 0, \\ B_{y_1} - \lambda_2(L) &= \omega_4 - \lambda_2(L) = 0. \end{aligned}$$

Since these equations can all be satisfied by appropriately choosing the ω parameters, we effectively eliminate the boundary conditions.

Control Equation

The control equation is derived from the partial derivative of H with respect to θ:

$$\begin{aligned} H_\theta &= -\lambda_1 \sin\theta + (sx + \lambda_2)\cos\theta, \\ &= -s(c_y - y)\sin\theta + s(x - c_x)\cos\theta = 0. \end{aligned}$$

This eliminates $\lambda_1(t)$ and $\lambda_2(t)$ entirely. The reduced system is a Boundary Value Problem (BVP) that follows directly.

Reduced BVP

The resulting BVP is:

$$\begin{cases} x' = \cos\theta, \\ y' = \sin\theta, \\ 0 = (x - c_x)\cos\theta + (y - c_y)\sin\theta, \\ x(0) = b, \quad y(0) = 0, \quad x(L) = a, \quad y(L) = 0. \end{cases}$$

Geometric Interpretation

Define $X = x - c_x$ and $Y = y - c_y$. The third equation becomes:

$$\begin{aligned} 0 &= (x - c_x)x' + (y - c_y)y', \\ &= XX' + YY', \\ &= \frac{1}{2}\frac{d}{dt}\left(X^2 + Y^2\right). \end{aligned}$$

Integrating this relation shows that:

$$R^2 = X^2 + Y^2 = (x - c_x)^2 + (y - c_y)^2,$$

which is the equation of a circle with radius R, centered at (c_x, c_y).

The trajectory described by $(x(t), y(t))$ is constrained to lie on a circle of radius R centered at (c_x, c_y). The constansts are otained by matching the boundary conditions.

5.3 An Optimal Control Problem with Path Constraints

The following problem involves a linear control which is bounded, and a constraint on both state and control. The formulation is

$$\begin{aligned} \text{minimize:} \quad & \int_0^3 (t-4)u \, \mathrm{d}t, \\ \text{subject to the dynamic:} \quad & x' = u, \\ \text{constrained by:} \quad & x(0) = 0, \quad x(3) = 3, \quad 0 \le u \le 2, \\ \text{and:} \quad & g(t,x,u) = x - t - u + 1 \le 0. \end{aligned}$$

The Hamiltonian for the problem is given by:

$$\begin{aligned} \mathcal{H} &= (t-4)u + \lambda u + \mu(x - t - u + 1) \\ &= (t - 4 + \lambda - \mu)u + \mu(x - t + 1). \end{aligned}$$

At $t = 0$, we find that the constraint g is satisfied only if $g(0) = 1 - u(0) \le 0$. This implies that the initial control must be $u(0) \ge 1$.

From the Hamiltonian, we derive the equations necessary to form the boundary value problem:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial u} &= t - 4 + \lambda - \mu = 0, \\ \lambda' &= -\frac{\partial \mathcal{H}}{\partial x} = -\mu, \quad \mu \ge 0. \end{aligned}$$

The multiplier of the constraint, μ, should be non-negative when the bound is active (i.e., when $g = 0$) and should equal zero when the bound is inactive.

Suppose the constraint is inactive, which implies $\mu = 0$. In this case, the boundary value problem yields $\lambda = c$, where c is a constant derived from the equation $\frac{\partial \mathcal{H}}{\partial x}$. However, from the equation $\frac{\partial \mathcal{H}}{\partial u}$, we get $\lambda = 4 - t$.

This leads to a contradiction: there cannot be an arc where the constraint is inactive. Therefore, on the interval $[0, 3]$, the constraint must be sharp. By combining the two equations $\lambda' = -\mu$ and $t - 4 + \lambda - \mu = 0$, we obtain the differential equation:

$$t - 4 + \lambda + \lambda' = 0.$$

Its solution is $\lambda(t) = \alpha \mathrm{e}^{-t} - t + 5$ for an unknown constant α. From the multiplier we can get $\mu = -\lambda' = \alpha \mathrm{e}^{-t} + 1$ which is non negative on $[0, 3]$ for $\alpha \ge \mathrm{e}^3 \approx 20.08553692$.

The control is given differentiating $g = 0$ with the combination of $x' = u$, that is $x' - 1 - u' = u - u' - 1 = 0$. The initial value of the control $u(0) = 1$ allows to solve the differential equation yielding $u(t) = 1$. Now from $g = 0$ we obtain $x(t) = t$.

The presence of a minimum is given by the application of the Weierstrass condition $\mathcal{H}(u) - \mathcal{H}(u^\star) > 0$, where $u^\star$ is the optimal control and $u > 1$ is another admissible control.

Observing that $\mu g = 0$ the Weierstrass condition must be checked only for $(t - 4 + \lambda)(u - u^\star) > 0$. Notice that $(t - 4 + \lambda)$ is strictly positive for $\alpha \ge \mathrm{e}^3$ and $u > u^\star = 1$, see Figure 5.6.

5.4 Dubins Car

The Dubins car model represents a simple car with three degrees of freedom, conceptualized as a rigid body moving in a two-dimensional plane. In this model, the back wheels can only slide, which complicates maneuvers like parallel parking.

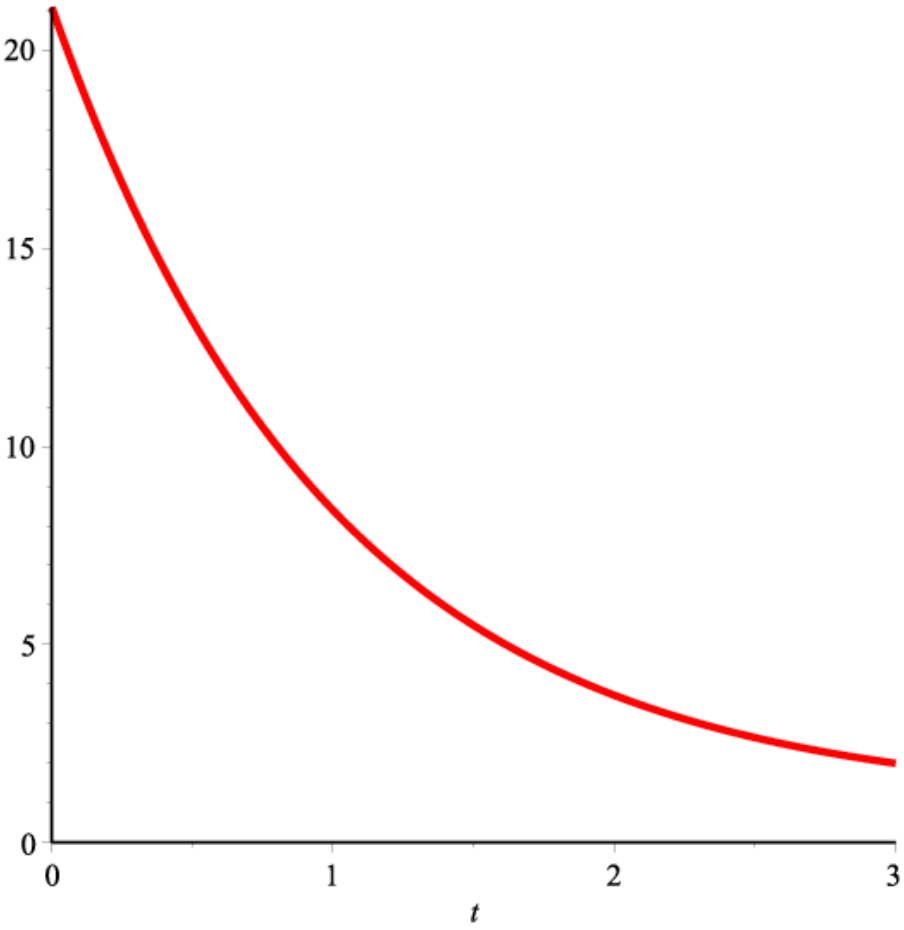


Figure 5.6: The plot of $t - 4 + \lambda$.

If all wheels could rotate freely, parking would be a straightforward task.

The position of the car is described by the triplet $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}$, where x and y represent the coordinates in the plane, and θ denotes the angle of the car relative to the x-axis.

From the geometry of this model (with certain constants simplified to unity), we can derive the following system of differential equations:

$$\begin{aligned} x' &= \cos(\theta), \\ y' &= \sin(\theta), \\ \theta' &= u. \end{aligned}$$

The objective is to drive the car from a specified position to the origin in minimum time. The control variable u, which represents the steering angle rate change, is constrained to lie within the interval $[-2, 2]$. Therefore, the optimal control problem can be formulated as follows:

$$\begin{aligned} \min T = \min \int_0^T 1 \, \mathrm{d}t \qquad & s.t. \qquad |u| \leq 2 \qquad \text{with} \\ x' = \cos(\theta), \qquad & x(0) = 4, \qquad x(T) = 0, \\ y' = \sin(\theta), \qquad & y(0) = 0, \qquad y(T) = 0, \\ \theta' = u, \qquad & \theta(0) = \frac{\pi}{2}. \end{aligned}$$

The control variable u appears linearly in the equations, indicating the potential for a singular arc in the solution. The Hamiltonian for this problem is given by:

$$\mathcal{H}(x, y, \theta, \lambda_1, \lambda_2, \lambda_3, u) = 1 + \lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) + \lambda_3 u.$$

From the PMP,

$$u = \underset{u \in [-2,2]}{\operatorname{argmin}} \mathcal{H}(x, y, \theta, \lambda_1, \lambda_2, \lambda_3, u) = u = \underset{u \in [-2,2]}{\operatorname{argmin}} \lambda_3 u$$

therefore we can write

$$u = \begin{cases} 2 & \text{if} \quad \lambda_3 < 0, \\ ? & \text{if} \quad \lambda_3 = 0, \\ -2 & \text{if} \quad \lambda_3 > 0. \end{cases}$$

The equation of the costate are derived from the Hamiltonian,

$$\begin{aligned} -\lambda_1' &= \frac{\partial \mathcal{H}}{\partial x} = 0, \\ -\lambda_2' &= \frac{\partial \mathcal{H}}{\partial y} = 0, \\ -\lambda_3' &= \frac{\partial \mathcal{H}}{\partial \theta} = -\lambda_1 \sin\theta + \lambda_2 \cos\theta. \end{aligned}$$

From the previous equations, the multipliers λ_1 and λ_2 are real constants. Upon further differentiation of λ_3', we find that within the singular arc, $\lambda_3'' = 0$. Therefore, we can conclude that:

$$\lambda_3'' = \lambda_1 \theta' \cos\theta + \lambda_2 \theta' \sin\theta = \lambda_1 u \cos\theta + \lambda_2 u \sin\theta = 0,$$

In the singular arc, we have the condition $u(\lambda_1 \cos\theta + \lambda_2 \sin\theta) = 0$, which implies that $u = 0$. Given this information about u, we can reconstruct the singular arc. Since $\theta' = u = 0$, it follows that $\theta(t) = K$, where K is a constant. Consequently, this leads to:

$$x' = \cos K \quad \text{and} \quad y' = \sin K.$$

Integrating these equations yields the singular arc:

$$x(t) = t\cos K + K_1,$$
$$y(t) = t\sin K + K_2,$$

where K_1 and K_2 are two real constants.

Next, we analyze the non-singular arc, where we have:

$$\theta(t) = -2\,t\,\text{sign}(\lambda_3) + k_3.$$

Using the initial condition $\theta(0) = \frac{\pi}{2}$, we can solve for k_3:

$$k_3 = \frac{\pi}{2}.$$

Letting $m = -2\,\text{sign}(\lambda_3)$, the associated arc becomes:

$$x'(t) = \cos(mt + k_3) \quad \Longrightarrow \quad x(t) = \frac{1}{m}\sin(mt + \pi/2) + k_1,$$
$$y'(t) = \sin(mt + k_3) \quad \Longrightarrow \quad y(t) = -\frac{1}{m}\cos(mt + \pi/2) + k_2.$$

We now assume that there is only a singular arc in the interval $(t_A, T]$, where t_A represents the unknown switching instant. This assumption is supported by the fact that the final condition for θ is free, leading to the conclusion that the associated multiplier λ_3 is zero at $t = T$.

Consequently, the arc is non-singular in the interval $[0, t_A]$. By examining the initial conditions, we can infer that $\lambda_3 < 0$ in this interval. This results in:

$$k_1 = 4 - \frac{1}{2} = \frac{7}{2} \quad \text{and} \quad k_2 = -\cos\left(\frac{\pi}{2}\right) = 0.$$

Thus, the associated trajectory is given by:

$$x(t) = \frac{1}{2}\cos(2t) + \frac{7}{2},$$
$$y(t) = \frac{1}{2}\sin(2t),$$
$$\theta(t) = 2t + \frac{\pi}{2}.$$

We can now connect the first arc with the second, as the trajectory is a continuous function at $t = t_A$. This ensures a smooth transition between the two arcs.

$$x(t_A) = t_A\cos K + K_1 = \frac{1}{2}\cos(2t_A) + \frac{7}{2},$$
$$y(t_A) = t_A\sin K + K_2 = \frac{1}{2}\sin(2t_A),$$
$$\theta(t_A) = K = 2t_A + \frac{\pi}{2}.$$

We can also impose the final conditions

$$x(T) = T\cos K + K_1 = 0,$$
$$y(T) = T\sin K + K_2 = 0.$$

This forms a nonlinear system of five equations involving the five unknowns K, K_1, K_2, t_A, and T. We will solve this system next.

By performing some algebraic manipulations, we can simplify the relationships among K, K_1, and K_2. This reduction leads us to two equations with two unknowns: T and t_A.

$$-T\sin(2t_A) + t_A\sin(2t_A) + \frac{1}{2}\cos(2t_A) + \frac{7}{2} = 0,$$
$$T\cos(2t_A) - t_A\cos(2t_A) + \frac{1}{2}\sin(2t_A) = 0.$$

By multiplying the first equation by $\cos(2t_A)$ and adding it to the second equation multiplied by $\sin(2t_A)$, we can combine these terms. Similarly, if we multiply the first equation by $\sin(2t_A)$ and add it to the second equation multiplied by $\cos(2t_A)$, we obtain:

$$\frac{1}{2} + \frac{7}{2}\cos(2t_A) = 0,$$
$$-T + t_A + \frac{7}{2}\sin(2t_A) = 0.$$

From this pair of expressions, it becomes straightforward to solve for the switching instant t_A and the final (minimum) time T. Consequently, we can also determine the three constants K, K_1, and K_2.

$$\begin{aligned}
t_A &= \frac{\pi}{2} - \frac{1}{2}\arccos\frac{1}{7} && \approx 0.857071948,\\
T &= \frac{\pi}{2} - \frac{1}{2}\arccos\frac{1}{7} + 2\sqrt{3} && \approx 4.321173564,\\
K &= \frac{3\pi}{2} - \arccos\frac{1}{7} && \approx 3.284940223,\\
K_1 &= \frac{2\sqrt{3}}{7}\left(\pi - \arccos\frac{1}{7} + 4\sqrt{3}\right) && \approx 4.276852666,\\
K_2 &= \frac{\pi}{14} - \frac{1}{14}\arccos\frac{1}{7} + \frac{2\sqrt{3}}{7} && \approx 0.6173105091.
\end{aligned}$$

These constants allow us to fully characterize the state of the system at any time $t \in [0, T]$. Next, we need to specify the costate. We have already established that λ_1 and λ_2 are constant. For λ_3, we have the differential equation given by

$$-\frac{\partial \mathcal{H}}{\partial \theta} = 0$$

for the singular arc. To set up a nonlinear system involving the three unknowns λ_1, λ_2, and λ_3, we require two additional equations. One of these can be derived from the Hamiltonian itself, which is autonomous and thus equal to zero. The third equation can be expressed as the multiplier in the interval $[0, t_A]$:

$$\lambda_3(t) = -\frac{\lambda_1}{2}\cos(2t + k_3) - \frac{\lambda_2}{2}\sin(2t + k_3) + \lambda_3(0).$$

Here, we introduce a fourth unknown, $\lambda_3(0)$, but we do not require an additional equation for it. Consequently, the nonlinear system at $t = t_A$ is given by:

$$\mathcal{H} = 1 + \lambda_1\cos(\theta) + \lambda_2\sin(\theta) + \lambda_3 u = 0,$$
$$\lambda_3'(t_A) = -\frac{\partial \mathcal{H}}{\partial \theta} = \lambda_1\sin\theta - \lambda_2\cos\theta = 0,$$
$$\lambda_3(t_A) = -\lambda_1/2\cos(2t_A + k_3) - \lambda_2/2\sin(2t_A + k_3) + \lambda_3(0).$$

The solution of the system gives the following expressions,

$$\lambda_1 = \frac{\cos(k_3)}{\sin(2t_A)\cos k_3 - \sin K \cos(2t_A)} = \frac{4}{7}\sqrt{3} \quad \approx 0.9897433188,$$
$$\lambda_2 = \frac{\sin(k_3)}{\sin(2t_A)\cos k_3 - \sin K \cos(2t_A)} = \frac{1}{7} \quad \approx 0.1428571429,$$
$$\lambda_3(0) = -\frac{1}{2}.$$

In Figure 5.7 the plots for state, costate, control and trajectory.

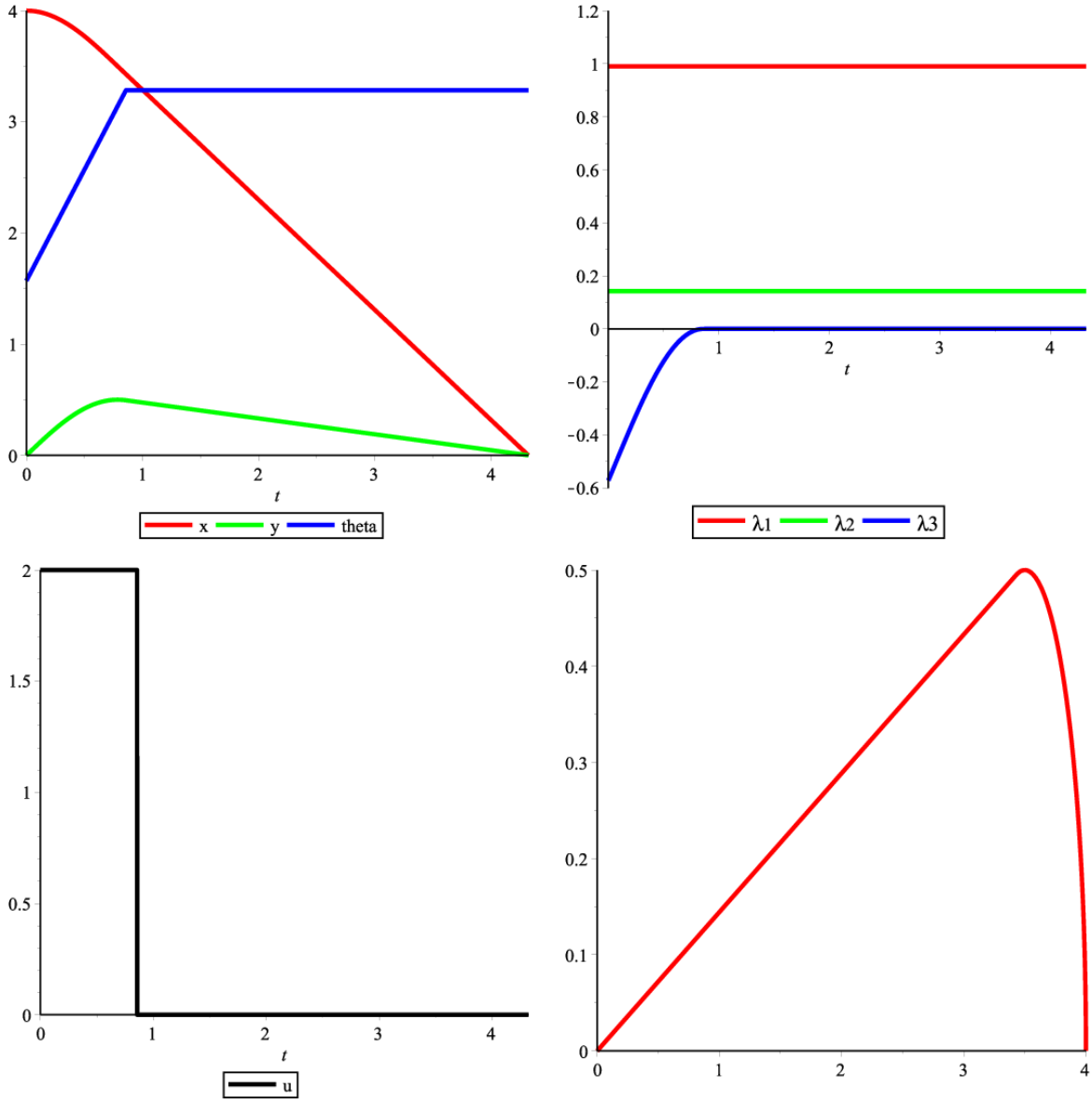


Figure 5.7: Variational solution to the Dubins car problem. On the top left the states , on the right the costates, below the control and the trajectory.

5.5 Four Singular Optimal Control Problems

In this chapter, we present four optimal control problems (OCPs) that feature singular arcs. Three of these examples are drawn from the PROPT handbook, which accompanies a com-

mercial software tool for solving OCPs.

Most of these problems are discussed in [1], though only a few provide exact solutions. Here, we present the analytical solutions to some of these problems, enabling a precise comparison with the numerical solutions generated by the software used for benchmarking.

Optimal Control Problem with Singular Controls

In optimal control theory, when the Hamiltonian $\mathcal{H}$ is linear in the control variable u, the solution may exhibit discontinuities. If the switching function associated with the control is not sustained over an interval—meaning the coefficient of u in $\mathcal{H}$ equals zero only at isolated points—the control is classified as bang-bang. A bang-bang control always takes on extreme values from the control set.

Conversely, if the coefficient of u in $\mathcal{H}$ is zero over an interval, the control is termed singular. In this case, the choice of u must be derived from information beyond Pontryagin's minimum principle. The moments when the optimal control transitions between states or switches to singular controls are referred to as switch times, which can be challenging to identify.

5.5.1 Singular 1

Consider the problem of minimizing the functional:

$$\begin{aligned} \text{minimize:} \quad & \int_{-1}^{1} (x - 1 + t^2)^2 \, \mathrm{d}t, \\ \text{subject to:} \quad & x' = u, \\ \text{with control bounded by:} \quad & |u| \leq 1. \end{aligned}$$

The Hamiltonian for this problem is given by:

$$\mathcal{H} = (x - 1 + t^2)^2 + \lambda u + p_\varepsilon(u),$$

where $p_\varepsilon(u)$ is a penalty function designed to manage constrained controls within the interval $[-1, 1]$. An example of such a function is

$$p_\varepsilon(u) = -\varepsilon \ln\left(\cos\left(\frac{\pi}{2}u\right)\right).$$

As $\varepsilon \to 0$, this function approaches zero within the interval $[-1, 1]$ and increases to infinity outside that range. Consequently, the Pontryagin minimum principle becomes ineffective in this scenario.

The stationarity condition for the optimal control problem is:

$$\frac{\partial \mathcal{H}}{\partial u} = \lambda + \frac{\partial p_\varepsilon(u)}{\partial u} = 0.$$

This leads to:

$$\begin{aligned} \lambda(t) &= -\varepsilon \frac{\pi}{2} \tan\left(\frac{\pi}{2}u(t)\right) \implies \\ u_\varepsilon(t) &= -\frac{2}{\pi} \arctan\left(\frac{2\lambda(t)}{\varepsilon\pi}\right) \xrightarrow[\varepsilon \to 0]{} -\operatorname{sign}(\lambda(t)), \qquad \lambda(t) \neq 0. \end{aligned}$$

Thus, when $\lambda \neq 0$, the control takes values $u = \pm 1$. When $\lambda = 0$, the control is singular.

The state is not specified at the boundaries, leading to the transversality conditions $\lambda(-1) = \lambda(1) = 0$. Differentiating the Hamiltonian with respect to x provides information about the multiplier λ:

$$\lambda' = -\frac{\partial \mathcal{H}}{\partial x} = -2(x - 1 + t^2).$$

Integrating this expression yields:

$$\lambda(t) = -2 \int_{-1}^{t} (x(s) - 1 + s^2)\, \mathrm{d}s + k, \tag{5.10}$$

where $k \in \mathbb{R}$ can be determined using the initial condition $\lambda(-1) = 0$:

$$0 = \lambda(-1) = -2 \int_{-1}^{-1} (x(s) - 1 + s^2)\, \mathrm{d}s + k \implies k = 0.$$

Substituting $k = 0$ into Equation 5.10 allows us to write:

$$0 = \lambda(1) = -2 \int_{-1}^{1} (x(s) - 1 + s^2)\, \mathrm{d}s \implies$$

$$\int_{-1}^{1} (x(s) - 1 + s^2)\, \mathrm{d}s = 0.$$

For t in the interval $[t_A, t_B] \subset [-1, 1]$, where $\lambda(t) = 0$, we have:

$$\begin{aligned} 0 &= \lambda(t) \\ &= -2 \int_{-1}^{t_A} (x(s) - 1 + s^2)\, \mathrm{d}s - 2 \int_{t_A}^{t} (x(s) - 1 + s^2)\, \mathrm{d}s \\ &= \lambda(t_A) - 2 \int_{t_A}^{t} (x(s) - 1 + s^2)\, \mathrm{d}s. \end{aligned} \tag{5.11}$$

Differentiating the right-hand side gives:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \left(-2 \int_{t_A}^{t} (x(s) - 1 + s^2)\, \mathrm{d}s \right)$$

$$\Downarrow$$

$$x(t) - 1 + t^2 = 0, \quad t \in [t_A, t_B].$$

Thus, when the control is singular, we find that

$$x(t) = 1 - t^2,$$

and from the dynamics of the problem,

$$u(t) = -2t \quad \text{for } t \in [t_A, t_B].$$

It is important to note that for $|t| > 1/2$, the control $u(t) = -2t$ does not satisfy the imposed bounds. Therefore, we need to identify two switching times, t_A and t_B, such that $x' = u$ and $|u| \leq 1$. For $t \in [-1, t_A)$, we have $u(t) = 1$, and for $t \in (t_B, 1]$, we have $u(t) = -1$. The corresponding state equations are then given by:

- For $t < t_A$: $x(t) = t + a$
- For $t > t_B$: $x(t) = -t + b$.

From Equation 5.11 and the fact that for $t \in [t_A, t_B]$, $x(t) = 1 - t^2$, it follows that $\lambda(t_A) = 0$, i.e,

$$\begin{aligned} 0 = \lambda(t_A) &= -2 \int_{-1}^{t_A} (x(s) - 1 + s^2)\, \mathrm{d}s \\ &= -2 \int_{-1}^{t_A} (s + a - 1 + s^2)\, \mathrm{d}s \\ &= -2 \left[\frac{1}{3}(t_A + 1) \left(t_A^2 + \frac{1}{2} t_A - \frac{7}{2} + 3a \right) \right]. \end{aligned}$$

This expression is equal to zero for

$$a = -\frac{1}{3}t_A^2 - \frac{1}{6}t_A + \frac{7}{6} \implies x(t) = t - \frac{1}{3}t_A^2 - \frac{1}{6}t_A + \frac{7}{6}. \tag{5.12}$$

Now, because x is continuous, in the switching point must hold $t_A + a = 1 - t_A^2$. From that equation it is possible to solve $t_A = -\frac{1}{4}$ and $a = \frac{19}{16}$.

With similar consideration it is possible to solve b and t_B: in fact,

$$0 = \lambda(1) = \lambda(t_B) - 2\int_{t_B}^{1} x(s) - 1 + s^2 \,\mathrm{d}s \implies$$
$$\int_{t_B}^{1} x(s) - 1 + s^2 \,\mathrm{d}s = 0.$$

Using the expression $x(t) = -t + b$ for $t \in (t_B, 1]$, the previous integral yields

$$\begin{aligned} 0 &= \int_{t_B}^{1} -s + b - 1 + s^2 \,\mathrm{d}s \\ &= -\frac{1}{3}(t_B - 1)\left(t_B^2 - \frac{1}{2}t_B - \frac{7}{2} + 3b\right), \end{aligned}$$

which gives the same result of Equation 5.12,

$$b = -\frac{1}{3}t_B^2 - \frac{1}{6}t_B + \frac{7}{6} \implies x(t) = -t - \frac{1}{3}t_B^2 - \frac{1}{6}t_B + \frac{7}{6}. \tag{5.13}$$

Again, for the continuity of x, $-t_B + b = 1 - t_B^2$, implies that $t_B = \frac{1}{4}$ and $b = \frac{19}{16}$. This also shows the symmetry of the solution. With this optimal control u and state x the minimum of the functional is

$$\begin{aligned} \int_{-1}^{1}(x - 1 + t^2)^2 \,\mathrm{d}t &= \int_{-1}^{-1/4}\left(t + \frac{19}{16} - 1 + t^2\right)^2 \mathrm{d}t \\ &\quad + \int_{1/4}^{1}\left(-t + \frac{19}{16} - 1 + t^2\right)^2 \mathrm{d}t \\ &= \frac{9}{1280} = 0.00703125. \end{aligned}$$

Collecting the three phases the analytical solution of the optimal control problem is,

$$x(t) = \begin{cases} t + a \text{ for} & t \in [-1, t_A) = \left[-1, -\frac{1}{4}\right) \\ 1 - t^2 \text{ for} & t \in [t_A, t_B] = \left[-\frac{1}{4}, \frac{1}{4}\right], \\ -t + b \text{ for} & t \in (t_B, 1] = \left(\frac{1}{4}, 1\right] \end{cases} \qquad a = b = \frac{19}{16},$$

$$u(t) = \begin{cases} 1 \text{ for} & t \in [-1, t_A) = \left[-1, -\frac{1}{4}\right) \\ -2t \text{ for} & t \in [t_A, t_B] = \left[-\frac{1}{4}, \frac{1}{4}\right] \\ -1 \text{ for} & t \in (t_B, 1] = \left(\frac{1}{4}, 1\right] \end{cases}$$

$$\lambda(t) = \begin{cases} -\frac{2}{3}t^3 - t^2 - \frac{3}{8}t - \frac{1}{24} \text{ for} & t \in [-1, t_A) = \left[-1, -\frac{1}{4}\right) \\ 0 \text{ for} & t \in [t_A, t_B] = \left[-\frac{1}{4}, \frac{1}{4}\right], \\ -\frac{2}{3}t^3 + t^2 - \frac{3}{8}t + \frac{1}{24} \text{ for} & t \in (t_B, 1] = \left(\frac{1}{4}, 1\right]. \end{cases}$$

A graphical representation of the state x, the control u and the costate λ is shown in Figure 5.8.

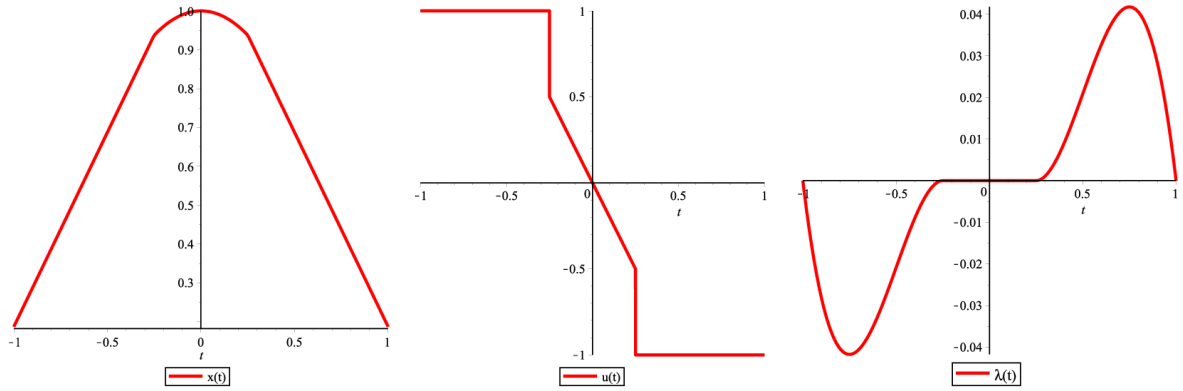


Figure 5.8: Variational solution to the singular arc problem. On the left the trajectory of the state $x(t)$, in the middle the control $u(t)$ and on the right the costate $\lambda(t)$.

In Figure 5.9 the numerical solution obtained with ACADO.

5.5.2 Singular 2

The first example is relatively straightforward from an analytical standpoint and could be approached using intuition alone. However, it has been explored by several researchers—including Jacobson, Chen, Huang, and Luus (refer to Example 10.2.1)—who have shown that achieving a reliable numerical solution is more challenging than it initially appears. The dynamical system is characterized by two equations and a functional:

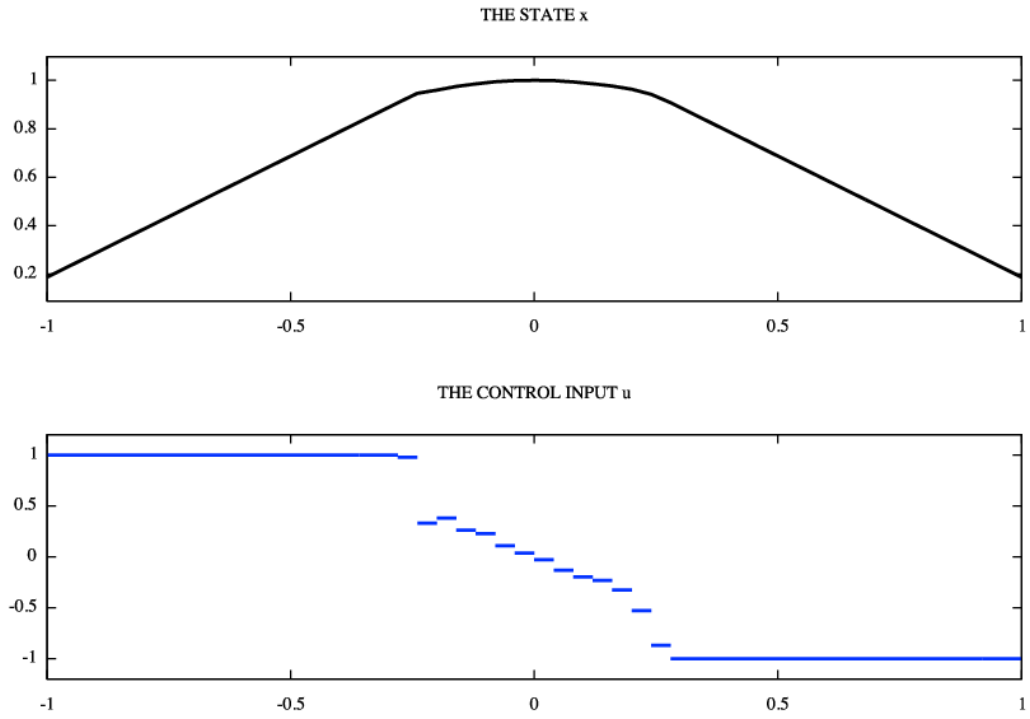


Figure 5.9: Numerical solution given by ACADO.

$$\begin{aligned} \text{minimize:} \quad & x_2(T) \\ \text{subject to the dynamic:} \quad & x_1(t)' = u(t), \\ & x_2(t)' = \frac{1}{2}x_1(t)^2, \\ \text{with control bounded by:} \quad & |u| \leq 1. \end{aligned}$$

The boundary conditions provided are $x_1(0) = 1$ and $x_2(0) = 0$, leading to the conditions $\lambda_1(T) = 0$ and $\lambda_2(T) = 1$, where T is the final time, set to $T = 2$. The Hamiltonian for this problem is given by:

$$\mathcal{H} = \lambda_1 u + \frac{1}{2}\lambda_2 x_1^2.$$

The equation of the costate are derived from the Hamiltonian,

$$-\lambda_1' = \frac{\partial \mathcal{H}}{\partial x_1} = -\lambda_2 x_1,$$
$$-\lambda_2' = \frac{\partial \mathcal{H}}{\partial x_2} = 0,$$

Thus, we have $\lambda_2(t) = 1$, which is constant. Assuming there is only one switching point, we can infer that the singular arc is the second one. By deriving the multiplier λ_1 in the singular part, we obtain the singular control. From the equation $\lambda_1' = -x_1$, we find that $\lambda_1'' = -x_1' = -u = 0$. This indicates that the control is zero within the singular arc.

Next, we can integrate the dynamical system to determine both the state and the costate. The resulting expressions are:

$$
\begin{aligned}
x_1(t) &= K, \\
x_2(t) &= \frac{1}{2}K^2 t + K_2, \\
\lambda_1(t) &= -Kt + 2K, \\
\lambda_2(t) &= 1,
\end{aligned}
$$

for some constants K and K_2 that need to be determined, and for $t \in [t_A, T]$, where t_A is the unknown switching time.

In the first arc, the control is derived using Pontryagin's minimum principle, resulting in a constant control of $u(t) = -1$ for $t \in [0, t_A)$. By applying the initial conditions, we can solve for the state, yielding:

$$
\begin{aligned}
x_1(t) &= -t + 1, \\
x_2(t) &= \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{2}t, \\
\lambda_1(t) &= \frac{1}{2}t^2 - t + \lambda_0, \\
\lambda_2(t) &= 1,
\end{aligned}
$$

for $t \in [0, t_A)$. By applying the continuity of the states at the switching point, we can establish a nonlinear system involving the unknowns K, K_2, λ_0, and t_A. This system yields two solutions: one lacks physical significance, while the other provides $t_A = 1$, $K = 0$, $K_2 = \frac{1}{6}$, and $\lambda_0 = \frac{1}{2}$. Consequently, the corresponding value for $x_2(T)$ is $\frac{1}{6}$. The plots of the system are illustrated in Figure 5.10.

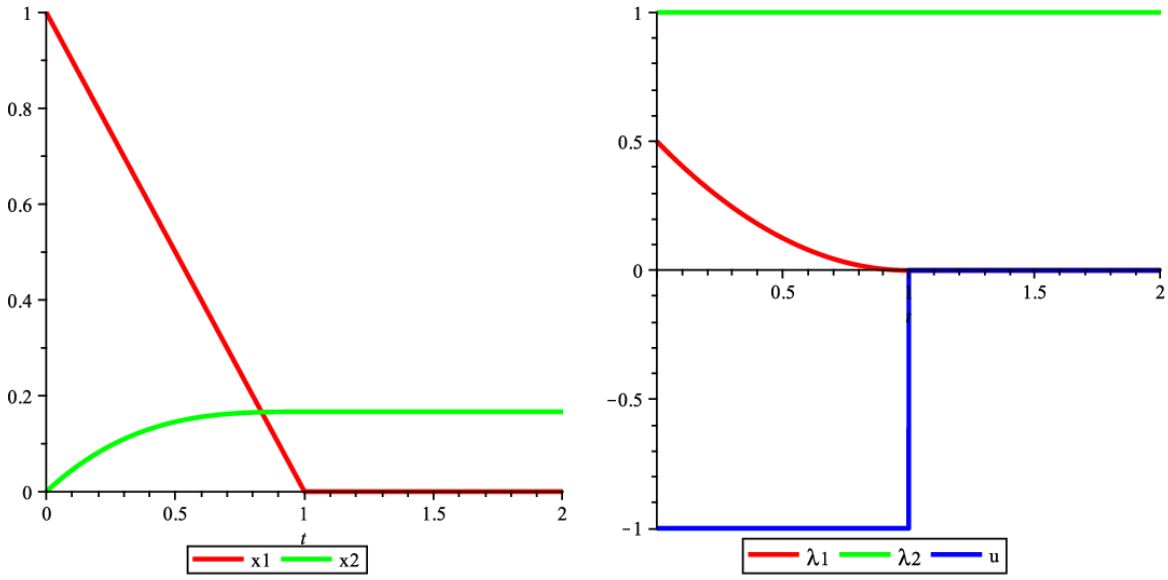


Figure 5.10: Variational solution to Singular Problem 1. On the left the states, on the right the costates and the control.

5.5.3 Singular 3

Constrained 1d vehicle

In this section, we analyze the model of a one-dimensional vehicle. We will explore various types of control strategies: a linear control based on jerk and a quadratic control on jerk. The linear control results in both singular and bang-bang arcs, while the quadratic control provides a smoother control profile.

The optimal control problem can be formulated as follows:

$$\min \int_0^T 1 + wJ^2 \, \mathrm{d}t$$

where w is set to zero in the first case, and $w = 0.1$ in the second case, subject to the following dynamic and path constraints:

$$\begin{aligned} s' &= v \\ v' &= a(p) \\ p' &= J \end{aligned}$$

with boundary conditions:

$$\begin{aligned} s(0) &= 0, & s(T) &= 200 \\ v(0) &= 10, & v(T) &= 10 \\ p(0) &= 0, & p(T) &= \text{free}. \end{aligned}$$

The path constraint is

$$a_{\min} - a(p) \leq 0, \qquad a(p) = p \cdot \begin{cases} a_1 & p \geq 0, \\ a_2 & p < 0. \end{cases}$$

where the constants are $a_{\min} = -5$, $a_1 = 3$ and $a_2 = 10$. The state p is constrained in $[-1, 1]$ and the control $J \in [-1, 1]$.

Jerk as quadratic control

5.5.4 Singular 4

Dadebo example 1

Here we analyze the problem proposed by many researchers which is stated in the paper by Dadebo et McAuley.

$$\text{minimize:} \quad y(1)$$

subject to the dynamical evolution

$$\begin{aligned} x' &= u \\ y' &= x^2 + u^2, \end{aligned}$$

with boundary conditions $x(0) = x(1) = 1$ and $y(0) = 0$.

The Hamiltonian for this problem is

$$\mathcal{H} = \lambda u + \mu(x^2 + u^2),$$

The adjoint equations are given by $\lambda' = -2\mu x$ and $\mu' = 0$, with the boundary condition $\mu(1) = 1$. The control is defined as $u = -\frac{\lambda}{2\mu}$. Consequently, the multiplier μ is constant, so we have $\mu(t) = 1$.

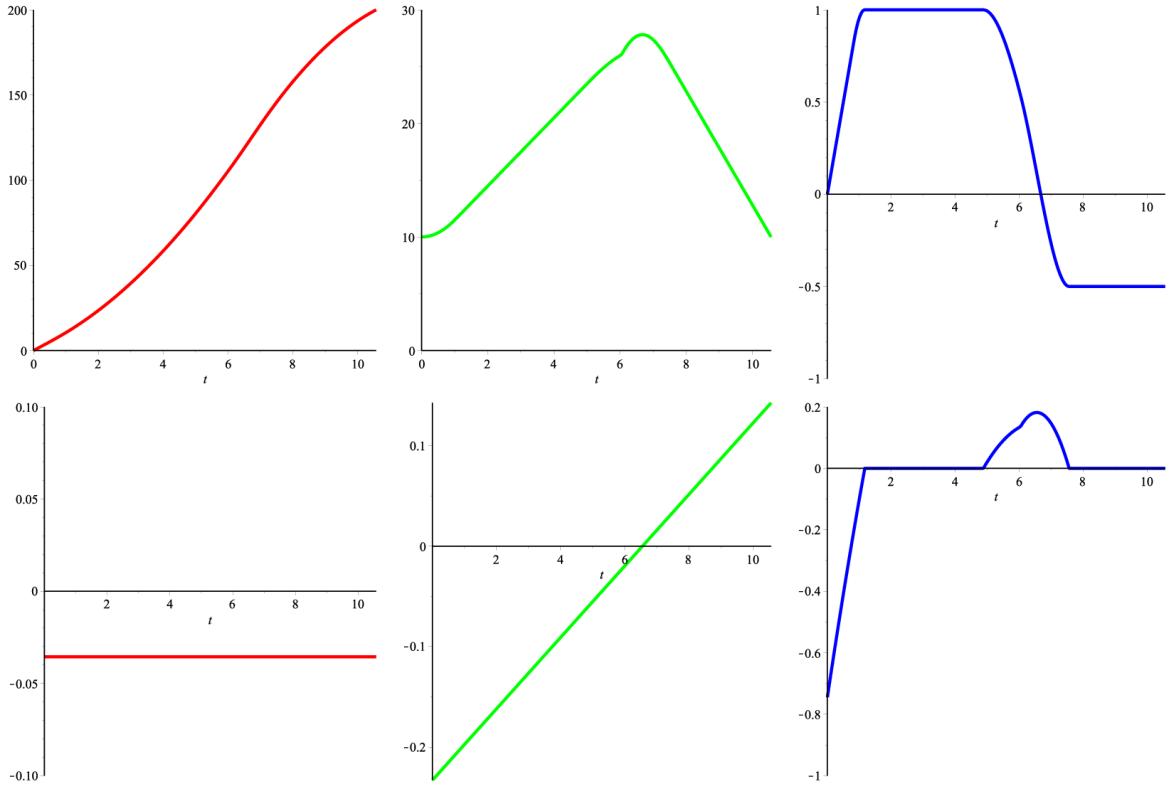


Figure 5.11: In red, green and blue respectively, on top the three states s, v and p, below the corresponding multipliers λ_1, λ_2, λ_3.}

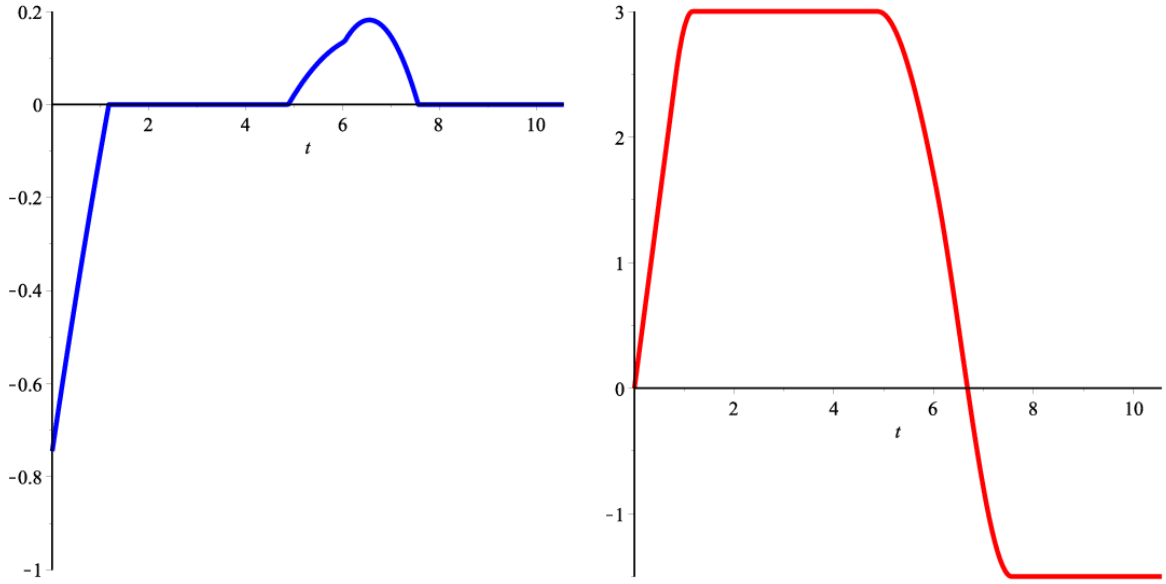


Figure 5.12: On the left the control J, on the right the acceleration $a(t)$.

Differentiating the equation for λ', we obtain $\lambda'' = -2x' = -2u = \lambda$. This leads to the characteristic equation, which has the general solution:

$$\lambda(t) = \alpha e^t + \beta e^{-t},$$

where α and β are unknown constants. From the multiplier λ, we can express the control as:

$$u(t) = -\frac{\alpha}{2}e^t - \frac{\beta}{2}e^{-t}.$$

Finally, the state x is determined by the differential equation $x' = u$, yielding:

$$x(t) = -\frac{\alpha}{2}e^t + \frac{\beta}{2}e^{-t} + c,$$

where c is a constant. The state y can be obtained from the equation $y' = x^2 + u^2$, which results in:

$$\begin{aligned} x^2 &= \frac{\alpha^2}{4}\mathrm{e}^{2t} + \frac{\beta^2}{4}\mathrm{e}^{-2t} + c^2 - \frac{\alpha\beta}{2} - \alpha c\mathrm{e}^t + \beta c\mathrm{e}^{-t} \\ u^2 &= \frac{\alpha^2}{4}\mathrm{e}^{2t} + \frac{\beta^2}{4}\mathrm{e}^{-2t} + \frac{\alpha\beta}{2} \\ x^2 + u^2 &= \frac{\alpha^2}{2}\mathrm{e}^{2t} + \frac{\beta^2}{2}\mathrm{e}^{-2t} + c^2 - \alpha c\mathrm{e}^t + \beta c\mathrm{e}^{-t}. \end{aligned}$$

Therefore, by integrating the last equation, we can compute the state y, which is given by:

$$y(t) = \frac{\alpha^2}{4}\mathrm{e}^{2t} - \frac{\beta^2}{4}\mathrm{e}^{-2t} + c^2 t - \alpha c\mathrm{e}^t - \beta c\mathrm{e}^{-t} + k$$

for an unknown constant k, which can be determined using the boundary conditions. We can find the four constants α, β, c, and k by solving the following nonlinear system:

$$\begin{aligned} x(0) &= -\frac{\alpha}{2} + \frac{\beta}{2} + c = 1 \\ x(1) &= -\frac{\alpha}{2}e + \frac{\beta}{2}\mathrm{e}^{-1} + c = 1 \\ y(0) &= \frac{\alpha^2}{4} - \frac{\beta^2}{4} - \alpha c - \beta c + k = 0 \\ \mathcal{H}(t=0) = \mathcal{H}(t=1) \quad &\Longrightarrow \quad -\alpha c + \beta c = -\alpha c e + \beta c\mathrm{e}^{-1}. \end{aligned}$$

The last equation arises from the constancy of the Hamiltonian along an optimal control and can be simplified to $c(\beta - \alpha) = c(\beta e^{-1} - \alpha e)$. Thus, if $c \neq 0$, we find that $\alpha = -\beta = 0$ and $c = 1$. Substituting these values into the third equation of the nonlinear system yields $k = 0$, resulting in one extremal solution given by $x(t) = 1$, $y(t) = t$, and $u(t) = 0$. Consequently, the minimized functional is $y(1) = 1$.

However, the nonlinear system also admits another solution when $c = 0$. From the first equation, we have $\beta = \alpha + 2$, which, when substituted into the second equation, gives $\alpha = \frac{-2}{1+e}$ and $\beta = \frac{2e}{1+e}$. Plugging these values into the third equation results in $k = \frac{e-1}{e+1}$.

This alternative solution yields a lower minimized functional compared to the previous one, achieving an optimal value of:

$$y(1) = 2\frac{e-1}{e+1} \approx 0.9242343144.$$

5.6 Free-Time Problems

Example 1

Given the free time optimal control problem

$$\begin{aligned} \text{minimize:} \quad & \int_0^T \left(1 + t^2\right) u^2 + y \, \mathrm{d}t \\ \text{subject to:} \quad & x' = y + u \qquad y' = xy \\ & x(0) = 0 \qquad y(T) = 1 \end{aligned}$$

the first thing to do is to rewrite the statement in a form where the integral bound are known by performing a change of variable ζ that we arbitrary set to be defined in the domain $[0, 1]$ resulting in the transformation $\zeta = t/T$. The optimal control problem can so be stated as

$$\begin{aligned} \text{minimize:} \quad & \int_0^1 \Big(\left(1 + \zeta^2 T(\zeta)^2\right) u^2(\zeta) + y(\zeta) \Big) T(\zeta) \mathrm{d}\zeta \\ \text{subject to:} \quad & \begin{cases} x'(\zeta) = T(\zeta)\big(y(\zeta) + u(\zeta)\big) \\ y'(\zeta) = T(\zeta)\, x(\zeta)\, y(\zeta) \\ T'(\zeta) = 0 \end{cases} \\ \text{and BC:} \quad & \begin{cases} x(0) = 0 \\ y(1) = 1 \end{cases} \end{aligned}$$

The normalized Hamiltonian is

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{\lambda}, u, \zeta) = T\left(1 + \zeta^2 T^2\right) u^2 + Ty + \lambda_1 T(y + u) + \lambda_2 Txy, \qquad \boldsymbol{q} = (x, y, T)^\mathsf{T}.$$

The canonical equations are

$$\begin{cases} x' = T(y + u), \\ y' = Txy, \\ T' = 0, \\ \lambda_1' = -\lambda_2 Ty, \\ \lambda_2' = -T - \lambda_1 T - \lambda_2 Tx, \\ \lambda_3' = -u^2(1 + 3\zeta^2 T^2) - y - \lambda_1(y + u) - \lambda_2 xy. \end{cases}$$

Since $x(1)$ and $y(0)$ are free, while $x(0) = 0$ and $y(1) = 1$ are fixed,

$$\lambda_1(1) = 0, \qquad \lambda_2(0) = 0.$$

If T is represented as a free constant state, both endpoint values of that state are free and

$$\lambda_3(0) = \lambda_3(1) = 0.$$

Stationarity with respect to the unconstrained control gives

$$2T\left(1 + \zeta^2 T^2\right) u + \lambda_1 T = 0,$$

hence, for $T > 0$,

$$u(\zeta) = -\frac{\lambda_1(\zeta)}{2\left(1 + \zeta^2 T^2\right)}.$$

The two conditions on λ_3 encode the stationarity of the free duration. Equivalently, T may be kept as a scalar optimization parameter rather than introduced as a state.

5.6.1 Example 2: Integral Constraint

Given the optimal control problem in the control u and states x, y

$$\begin{aligned} \text{minimize:} \quad & \int_0^1 \left(1 + x^2\right) u^2 \, \mathrm{d}t \\ \text{subject to:} \quad & x' = xy - u \\ & y' = x + u \\ & \int_0^1 \left(x + u\right) \mathrm{d}t = 2 \\ & y(0) = 2 \end{aligned}$$

the first thing to do is remove the integral constraint by adding a new state variable z such that

$$z' = x + u$$

with boundary conditions

$$z(0) = 0 \quad \text{and} \quad z(1) = 2$$

with that the problem becomes

$$\begin{aligned} \text{minimize:} \quad & \int_0^1 \left(1 + x^2\right) u^2 \, \mathrm{d}t \\ \text{subject to:} \quad & x' = xy - u \\ & y' = x + u \\ & z' = x + u \\ & y(0) = 2 \quad z(0) = 0 \quad z(1) = 2 \end{aligned}$$

We can so compute the Hamiltonian $\mathcal{H}$ and the endpoint Lagrangian $\mathcal{B}$ of the problem as

$$\begin{aligned} \mathcal{H}(\boldsymbol{q}, \boldsymbol{\lambda}, u, t), \qquad \boldsymbol{q} = (x, y, z)^\mathsf{T} &= \left(1 + x^2\right) u^2 + \lambda_1 \left(xy - u\right) + \lambda_2 \left(x + u\right) + \lambda_3 \left(x + u\right) \\ \mathcal{B}\left(y_0, z_0, z_1, \mu_1, \mu_2, \mu_3\right) &= \mu_1 \left(y_0 - 2\right) + \mu_2 z_0 + \mu_3 \left(z_1 - 2\right) \end{aligned}$$

The control obtained by the Pontryagin minimum principle is the solution of u that verifies $\frac{\partial \mathcal{H}}{\partial u} = 0$, and so we have

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial u} &= 2\left(1 + x^2\right) u - \lambda_1 + \lambda_2 + \lambda_3 \\ &\Downarrow \\ u(t) &= \frac{\lambda_1(t) - \lambda_2(t) - \lambda_3(t)}{2\left(1 + x(t)^2\right)} \end{aligned}$$

The additional adjoint ODE are described by the following derivatives:

$$\begin{aligned} \lambda_1' &= -\frac{\partial \mathcal{H}}{\partial x} = -2xu^2 - \lambda_1 y - \lambda_2 - \lambda_3 \\ \lambda_2' &= -\frac{\partial \mathcal{H}}{\partial y} = -\lambda_1 x \\ \lambda_3' &= -\frac{\partial \mathcal{H}}{\partial z} = 0 \end{aligned}$$

while the adjoint boundary conditions are

$$\begin{aligned}
&\lambda_1(0) = -\frac{\partial \mathcal{B}}{\partial x_0} = 0 \qquad \lambda_1(1) = \frac{\partial \mathcal{B}}{\partial x_1} = 0 \\
&\lambda_2(0) = -\frac{\partial \mathcal{B}}{\partial y_0} = -\mu_1 \quad \lambda_2(1) = \frac{\partial \mathcal{B}}{\partial y_1} = 0 \\
&\lambda_3(0) = -\frac{\partial \mathcal{B}}{\partial z_0} = -\mu_2 \quad \lambda_3(1) = \frac{\partial \mathcal{B}}{\partial z_1} = \mu_3
\end{aligned}$$

Some expression have been cancelled out because they are trivially solved for μ_i.
Thus the BVP become

$$\begin{cases}
\left.\begin{aligned} x' &= xy - u \\ y' &= x + u \\ z' &= x + u \end{aligned}\right\} & \text{original ODE} \\
\left.\begin{aligned} \lambda_1' &= -2xu^2 - \lambda_1 y - \lambda_2 - \lambda_3 \\ \lambda_2' &= -\lambda_1 x \\ \lambda_3' &= 0 \end{aligned}\right\} & \text{adjoint ODE} \\
\left.\begin{aligned} y(0) &= 2 \\ z(0) &= 0 \\ z(1) &= 2 \end{aligned}\right\} & \text{original BC} \\
\left.\begin{aligned} \lambda_1(0) &= 0 \\ \lambda_1(1) &= 0 \\ \lambda_2(1) &= 0 \end{aligned}\right\} & \text{adjoint BC} \\
u = \dfrac{\lambda_1 - \lambda_2 - \lambda_3}{2\left(1 + x^2\right)} & \text{Pontryagin}
\end{cases}$$

5.6.2 Example 3: Free Time with Two Controls

Given the free time optimal control problem in the controls u, w and states x, y

$$\begin{aligned}
\text{minimize:} \quad & \int_0^T \left(x^2 + y^2\right) \mathrm{d}t \\
\text{subject to:} \quad & x' = yu \\
& y' = x + w \\
& x(0) = 0 \quad y(0) = 0 \quad y(T) = 1 \\
& u, w \in [-1, 1]
\end{aligned}$$

the first thing to do is to transform the problem in fixed boundary optimal control problem by performing the change of variable $t = \zeta\, T(\zeta)$ (with $\zeta \in [0, 1]$) and so

$$\begin{aligned}
\text{minimize:} \quad & \int_0^1 T(\zeta)\left(x^2(\zeta) + y^2(\zeta)\right) \mathrm{d}\zeta \\
\text{subject to:} \quad & x'(\zeta) = T(\zeta) y(\zeta) u(\zeta) \\
& y'(\zeta) = T(\zeta)\Big(x(\zeta) + w(\zeta)\Big) \\
& T'(\zeta) = 0 \\
& x(0) = 0 \quad y(0) = 0 \quad y(1) = 1 \\
& u(\zeta), w(\zeta) \in [-1, 1]
\end{aligned}$$

From now on all the functions will be represented as depending from the independent variable ζ and it won't be written any more. With that said we can compute the Hamiltonian $\mathcal{H}$ and the endpoint Lagrangian $\mathcal{B}$ of the problem as

$$\begin{aligned}\mathcal{H}(\boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{u}) &= T(x^2 + y^2) + \lambda_1 T y u + \lambda_2 T(x + w),\\ \boldsymbol{q} &= (x, y, T)^\mathsf{T}, \qquad \boldsymbol{u} = (u, w)^\mathsf{T},\\ \mathcal{B}(x_0, y_0, y_1, \boldsymbol{\mu}) &= \mu_1 x_0 + \mu_2 y_0 + \mu_3(y_1 - 1).\end{aligned}$$

The solution of the controls obtained by the Pontryagin principle becomes

$$\begin{aligned}(u, w) &= \underset{(u,w)\in[-1,1]\times[-1,1]}{\operatorname{argmin}} \quad \mathcal{H}\left(\boldsymbol{q}, \boldsymbol{\lambda}, (u, w)^\mathsf{T}\right)\\ &= \underset{(u,w)\in[-1,1]\times[-1,1]}{\operatorname{argmin}} \quad T\left(x^2 + y^2\right) + \lambda_2 T x + T(\lambda_1 y u + \lambda_2 w)\\ &= \underset{(u,w)\in[-1,1]\times[-1,1]}{\operatorname{argmin}} \quad T(\lambda_1 y u + \lambda_2 w)\\ &= \underset{(u,w)\in[-1,1]\times[-1,1]}{\operatorname{argmin}} \quad \lambda_1 y u + \lambda_2 w\end{aligned}$$

to find the minimum you can use KKT

minimize

$$f(u, w) = \lambda_1 y u + \lambda_2 w$$

subject to

$$-1 \le u \le 1, \qquad -1 \le w \le 1,$$

Lagrangian

$$\mathcal{L}(u, w, \mu_1, \mu_2, \mu_3, \mu_4) = \lambda_1 y u + \lambda_2 w - \mu_1(1 - u) - \mu_2(1 + u) - \mu_3(1 - w) - \mu_4(1 + w)$$

KKT System for the Control Minimization

$$\begin{cases} \partial_u \mathcal{L} = \lambda_1 y + \mu_1 - \mu_2 = 0\\ \partial_w \mathcal{L} = \lambda_2 + \mu_3 - \mu_4 = 0\\ \mu_1(1 - u) = 0\\ \mu_2(1 + u) = 0\\ \mu_3(1 - w) = 0\\ \mu_4(1 + w) = 0 \end{cases} \tag{5.14}$$

The solution of Equation 5.14 is the following:

$$u(\zeta) = \begin{cases} -1, & \lambda_1 y > 0, \\ [-1,1], & \lambda_1 y = 0, \\ +1, & \lambda_1 y < 0 \end{cases}$$

$$w(\zeta) = \begin{cases} -1, & \lambda_2 > 0, \\ [-1,1], & \lambda_2 = 0, \\ +1, & \lambda_2 < 0 \end{cases}$$

remember that must be $T > 0$ for $T \leq 0$ solution is discarded.

With that stated we can build the boundary value problem solution of the free time optimal control problem by adding at the initial ODEs and boundary conditions the adjoint ones and the control law (related to Pontryagin minimum principle):

$$\begin{cases} \left.\begin{aligned} x' &= Tyu \\ y' &= T(x+w) \\ T' &= 0 \end{aligned}\right\} \text{ original ODE} \\ \left.\begin{aligned} \lambda_1' &= -\frac{\partial \mathcal{H}}{\partial x} = -2xT - \lambda_2 T \\ \lambda_2' &= -\frac{\partial \mathcal{H}}{\partial y} = -2Ty - \lambda_1 Tu \\ \lambda_3' &= -\frac{\partial \mathcal{H}}{\partial T} = -x^2 - y^2 - \lambda_1 yu - \lambda_2(x+w) \end{aligned}\right\} \text{ adjoint ODE} \\ \left.\begin{aligned} x(0) &= 0 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned}\right\} \text{ original BC} \\ \left.\begin{aligned} \lambda_1(1) &= 0 \\ \lambda_3(0) &= 0 \\ \lambda_3(1) &= 0 \end{aligned}\right\} \text{ adjoint BC} \\ \left.\begin{aligned} u &= \begin{cases} -1, & \lambda_1 y > 0, \\ [-1,1], & \lambda_1 y = 0, \\ 1, & \lambda_1 y < 0 \end{cases} \\ w &= \begin{cases} -1, & \lambda_2 > 0, \\ [-1,1], & \lambda_2 = 0, \\ 1, & \lambda_2 < 0 \end{cases} \end{aligned}\right\} \text{ Pontryagin} \end{cases}$$

The solutions of the boundary value problem are in the independent variable ζ, so to have the final solution in the time variable t every control/state law must be evaluated in $t = \zeta\, T(\zeta)$.

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